mathematics and statistics

(BOOK I. GEOMETRY-STATISTICS)

for higher secondary course

N. K. Shah,
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PLEDGE

"India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone lies my happiness."

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MATHEMATICS AND STATISTICS

FOR

HIGHER SECONDARY COURSE

(STD. XI)

(Book I—Geometry, Trigonometry, Statistics)

BY

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SYLLABUS OF MATHEMATICS AND STATISTICS
STANDARD XI

PAPER I

Geometry, Trigonometry and Statistics

SECTION I

1. Vectors : Vector as a directed line segment. Position vector of a point. Addition and subtraction of vectors. Multiplication of a vector by a scalar. Section formula. Collinear points and concurrent lines. Simple applications to geometry.

2. Co-ordinate Geometry: Rectangular Cartesian co-ordinate system. Cartesian and polar co-ordinates of a point. Interpretation of position vector as an ordered pair. Distance between two points. Co-ordinates of a point of division. Equation to a locus satisfying a given condition. Change of origin. Derivation of equation to a line in the following standard forms :

   (i) $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

   (ii) $y-y_1=m(x-x_1)$

   (iii) $y=mx+c$

   (iv) $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$

   (v) $\frac{x}{a}+\frac{y}{b}=1$ (vi) $x\cos\alpha+y\sin\alpha=p$

   The linear equation $ax+by+c=0$.

   Angle between two lines. Condition that the two lines be perpendicular and parallel. Intersection of two lines. Distance of a point from a line.

3. Solid Geometry and Mensuration : Parallelopipeds and pyramids, their surface area and volumes. Cylinders and Cones. Surface areas and volume of right circular cylinder and right circular cone. Cavalieri's principle. Volume of prisms. Sphere and spherical cap. Spherical segment.
   (Emphasis is on geometrical properties and mensuration)

SECTION II

4. Trigonometry: Angles and their measures—sexagesimal and radian. Circular functions—their domain, range and periodicity. Graphs of circular functions. Simple trigonometrical identities. Derivation of cos (A – B) from distance formula. Derivation of the formulae : cos (A + B), sin (A ± B), sin 2A, cos 2A, sin A/2, cos A/2. Standard formulae for conversion of sums into products and products into sums. Sine and cosine law for a triangle. Application to problems on heights and distances.

5. Statistics : Population. Frequency distribution and frequency diagrams. Ungrouped and grouped data, Averages—their merits and demerits. Measures of location and dispersion. Deviation from an average value. Absolute deviation from the mean and standard deviation. Variance.

PREFACE

We have great pleasure in presenting this concise text book on Mathematics for the Eleventh Standard. The book covers the revised syllabus prescribed by the Maharashtra State Board of Secondary Education—Poona.

We have endeavoured to introduce the new concepts in Mathematics in a simplified manner. Numerous solved examples and exercises (including objective types of questions) given in the book will, it is hoped, help the student in developing the foundation that is needed for further study.

Teachers as well as students are earnestly requested to make their suggestions and comments designed to enhance the utility of the book.

AUTHORS

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CHAPTER 1

Vectors

A very well known puzzle in mathematics is to ask that, if two trains leave Bomay, one at 8·00 a. m. and another at 9·00 a. m. with speeds of 30 m. p. h. and 60 m. p. h. respectively, which train will reach Poona first, given that the distance between Bombay and Poona is 120 miles. If innocently we say that the second train will reach Poona first since it takes only 2 hours for the journey as compared to the 4 hours required for the first, we are working under the assumption that both these trains are going towards Poona. This is a wrong assumption since nothing of the kind is given. Hence knowing the speed of the train is not enough unless we know its direction too.

There are many physical quantities which cannot be understood unless we specify magnitudes as well as directions. As against this there are other quantities like temperature, mass, volume which have only magnitude and no direction. Hence we distinguish between these two types of quantities by giving the following definitions.

§ 1·1. Scalars and vectors:

Definition (i): A physical quantity having magnitude only is called a *scalar.*

A scalar is just a number and has no direction, e.g. mass, volume, density, temperature are scalars.

Definition (ii) : A physical quantity which can be completely specified by stating both its magnitude and direction is called a *vector*, e. g. velocity, acceleration. displacement, force, electric current are vectors.

Though all these are vectors, a very important vector is displacement. For understanding the operations with vectors, it is beneficial if we consider a vector as a displacement.

The idea of identifying a vector with displacement immediately helps in representing a vector by a directed line segment as follows:

§ 1·11. Directed line segments:

Consider two points A and B. If a particle at A is displaced to the point B along the line AB then this displacement is written as $\overline{AB}$ where the displacement takes place in the direction from A to B.

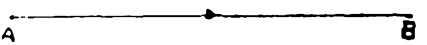


Fig. 1·1

Similarly the displacement of a particle from B to A shall be denoted as $\overline{BA}$. Obviously, these two displacements have the same magnitude but their directions are opposite and hence they represent different vectors.

In fact any vector can be represented by a directed line segment like this. This is done as follows:

We draw a line segment in the same direction as the vector and the length of this segment will be drawn so as to represent the magnitude of the vector on some scale.

Suppose we want to draw directed line segments representing the vectors: a velocity of 30 km. p. h. from North to South, a velocity of 60 km. p. h. from South to North and a velocity of 50 km. p. h. from East to West, then we proceed as follows:

Consider the scale in which 1 cm. represents a velocity of 10 km. p. h. Then the magnitudes of the three velocities of 30 km. p. h., 60 km. p. h. and 50 km. p. h. are represented by lengths of 3 cms., 6 cms. and 5 cms. respectively.

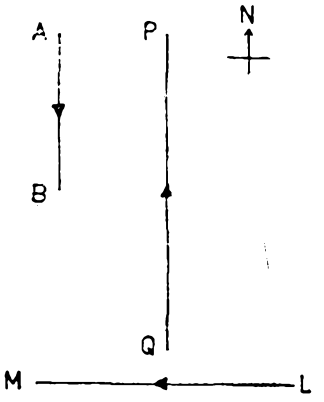


Fig. 1·2

Then if the direction of North is vertically upwards as shown in Fig. 1·2, the line segment $\overline{AB}$ of length 3 cms. represents the velocity of 30 km. p. h. in the direction from North to South.

Similarly, the line segment $\overline{QP}$ of length 6 cms. represents the velocity of 60 km. p. h. in the direction from South to North and the line segment $\overline{LM}$ of length 5 cms. represents the velocity of 50 km. p. h. in the East to West direction.

In this way a vector can always be represented by a directed line segment. We shall always think of a vector in terms of a directed line segment only.

§ 1·12. Types of vectors :

(1) Zero vector : A vector whose magnitude is zero is called a *zero vector*, whatever its direction may be. In fact, only in the case of a zero vector, the direction is not important. This is because a zero vector shall obviously be represented by a point and a point has no direction.

(2) Unit vector : A *unit vector* in a given direction is a vector of magnitude one unit in that direction.

(3) Equal vectors : Two vectors are said to be *equal* if they have equal magnitudes and their directions are the same.

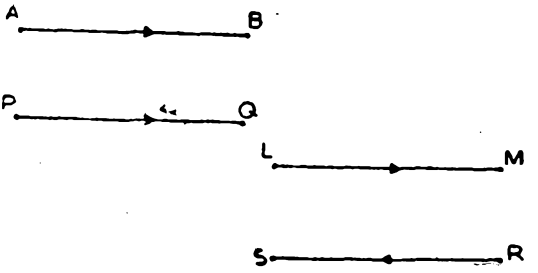


Fig. 1·3

In fig. 1·3 $\overline{AB} = \overline{PQ}$ but $\overline{LM} \neq \overline{RS}$ since their directions are opposite even though their magnitudes are equal.

§ 1·13. Addition of vectors : Consider fig 1·4. Suppose a particle is at A. If the particle undergoes first a displacement from A to B and next a displacement from B to C the final result is the same as if it had undergone a direct displacement from A to C, i. e. the effect produced by the vector $\overline{AB}$ together with the vector $\overline{BC}$ is the same as the effect of the vector $\overline{AC}$. Hence we have

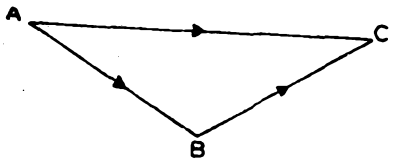


Fig 1·4

$$\overline{AB} + \overline{BC} = \overline{AC}$$

This law is called the *triangle law of vector addition.*

Note that the magnitude AC is not equal to the sum of the magnitudes of AB and BC. It is only the vector sum $\overline{AB} + \overline{BC}$ which is equal to the vector $\overline{AC}$, where magnitude $\overline{AC} = |\overline{AC}| = AC$.

Obviously $\overline{AB} + \overline{BA} = \overline{AA} = O$. $\therefore \overline{AB} = -\overline{BA}$.

Any two vectors can be added in this manner. Suppose $\bar{a}$ and $\bar{b}$ are two given vectors. Choose any point O. Draw $\overline{OA} = \bar{a}$, i.e. magnitude $OA = a$ and direction from O to A is the same as that of $\bar{a}$.

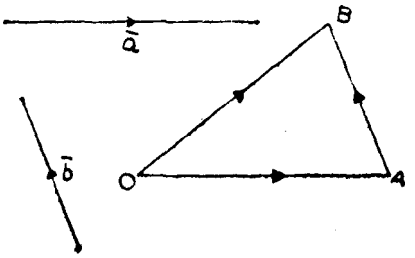


Fig. 1·5

Similarly, draw $\overline{AB} = \bar{b}$

Then $\bar{a} + \bar{b} = \overline{OA} + \overline{AB}$ $= \overline{OB}$. Hence $\overline{OB}$ is the sum of $\bar{a}$ and $\bar{b}$.

Parallelogram law : Consider two vectors $\overline{OA}$ and $\overline{OB}$ in fig. 1·6. Complete the parallelogram OACB. Then $\overline{AC} = \overline{OB}$.

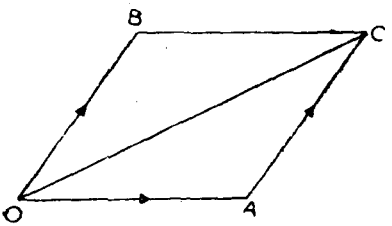


Fig. 1·6

$\therefore \overline{OA} + \overline{OB} = \overline{OA} + \overline{AC}$. $= \overline{OC}$.

Hence $\overline{OA} + \overline{OB} = \overline{OC}$. This can be written in words as " The sum of two vectors $\overline{OA}$ and $\overline{OB}$ is represented in magnitude and direction by the diagonal of the parallelogram through O of which OA and OB are adjacent sides.".

§ 1·14. Scalar multiplication :

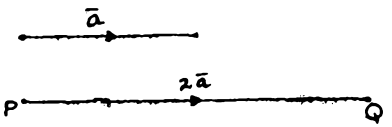


Fig. 1·7

Let $\bar{a}$ be a given vector

Let $\bar{a} = \overline{AB}$. If we draw the vector $\overline{PQ}$ such that direction of $\overline{PQ}$ is the same as that of

$\overline{AB}$ and $l(PQ) = 2 \cdot l(AB)$ then $\overline{PQ} = 2\overline{AB} = 2\bar{a}$. Hence $2\bar{a}$ is a vector whose direction is the same as $\bar{a}$ and whose magnitude is twice that of $\bar{a}$. In general we can say that if $\bar{a}$ is a vector and k is any scalar then

(i) if $k > 0$, $k \cdot \bar{a}$, is a vector whose magnitude is k times that of $\bar{a}$ and whose direction is the same as $\bar{a}$.

(ii) if $k = 0$, then $k \cdot \bar{a} = 0$, and

(iii) if $k < 0$ then $k \cdot \bar{a}$ is a vector whose magnitude is $|k|$ times of $\bar{a}$ and whose direction is opposite to that of $\bar{a}$.

This is called the *law of scalar multiplication.*

Specifically, $-\bar{b}$ will be a vector of magnitude equal to that of $\bar{b}$ and direction opposite to $\bar{b}$.

Hence the vector $\bar{a} - \bar{b}$ can be found as $\bar{a} + (-\bar{b})$ where the method of fig. 1·5 can be used to add $\bar{a}$ and $-\bar{b}$. The only change will be that we shall draw $\overline{OA} = \bar{a}$ and $\overline{AB}$ will be drawn in in the direction opposite to that of $\bar{b}$ and of the same length as $\bar{b}$.

Some simple laws followed by vector addition and scalar multiplication are given below:

If $\bar{a}, \bar{b}, \bar{c}$ are vectors and m, n are scalars, then

(1) $\bar{a} + \bar{b} = \bar{b} + \bar{a}$

(2) $\bar{a} + \bar{b} + \bar{c} = \bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$

(3) $m\bar{a} + n\bar{a} = (m + n)\bar{a}$

(4) $m\bar{a} + m\bar{b} = m(\bar{a} + \bar{b})$.

§ 1·15. Examples:

Ex. 1. If ABCD is a quadrilateral and M, N are the mid points of seg AB and seg CD respectively, then prove that

$$\overline{AD} + \overline{BC} = 2\overline{MN}$$

Ans. Obviously $\overline{AD} = \overline{AM} + \overline{MN} + \overline{ND}$ and

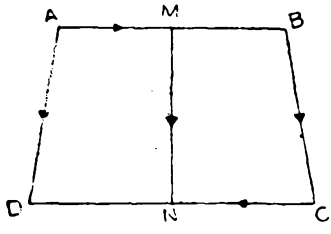


Fig. 1·8

$\overline{BC} = \overline{BM} + \overline{MN} + \overline{NC}$

$\therefore$ L. H. S. $= \overline{AD} + \overline{BC}$

$= \overline{AM} + \overline{MN} + \overline{ND}$

$+ \overline{BM} + \overline{MN} + \overline{NC}$

$= (\overline{AM} + \overline{BM}) + 2\overline{MN}$

$+ (\overline{ND} + \overline{NC})$

But $\overline{AM}$ and $\overline{BM}$ are equal in magnitude and opposite in direction.

$\therefore \quad \overline{AM} + \overline{BM} = 0$. Similarly $\overline{ND} + \overline{NC} = 0$.

$\therefore \quad$ L. H. S. $= 2\overline{MN}$ = R. H. S.

Ex. 2. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are vectors such that $\bar{a} + \bar{b} + \bar{c} = 0$, show that $\bar{a}$, $\bar{b}$, $\bar{c}$ represent the sides of a triangle taken in order.

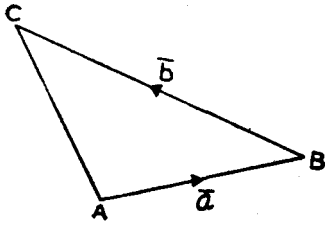


Fig. 1·9

Ans. Draw $\overline{AB} = \bar{a}$ and $\overline{BC} = \bar{b}$

Join AC $\quad \because \quad \bar{a} + \bar{b} + \bar{c} = 0$

$\therefore \quad \overline{AB} + \overline{BC} + \bar{c} = 0$

$\therefore \quad \overline{AC} + \bar{c} = 0$

$\therefore \quad \bar{c} = -\overline{AC}$

$\therefore \quad \bar{c} = \overline{CA}$

Hence, $\bar{a}$, $\bar{b}$, $\bar{c}$ represent $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ which are the sides of a triangle taken in order.

Ex. 3. ABCDEF is a regular hexagon. Show that

$$\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD}.$$

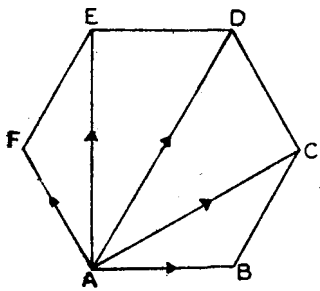


Fig. 1·10

Ans. In a regular hexagon, the sides AB and ED, BC and FE, CD and AF are parallel and equal. Hence $\overline{AF} = \overline{CD}$ and $\overline{AB} = \overline{ED}$. Putting these values in L. H. S. we get,

L. H. S.

$= \overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF}$

$= \overline{ED} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{CD}$

$= \overline{AD} + (\overline{AC} + \overline{CD}) + (\overline{AE} + \overline{ED})$

$= \overline{AD} + \overline{AD} + \overline{AD} = 3\overline{AD}$ = R. H. S.

Exercise 1·1 :

(1) ABCD is a quadrilateral and M and N are the mid points of AB and CD respectively. Show that $\overline{AC} + \overline{BD} = 2\overline{MN}$.

(2) ABCDE is a regular pentagon. Show that

$\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} = 2\overline{AC}$.

3) ABCD is a quadrilateral. Show that
$\overline{BA}+\overline{BC}+\overline{CD}+\overline{DA}=2\,\overline{BA}$.

(4) ABC is a triangle and P is a point on BC. If Q is a point such that $\overline{AP}+\overline{PB}+\overline{PC}=\overline{PQ}$, show that ABQC is a parallelogram.

§1·2. Position vector of a point :

Suppose O is a fixed point and P, Q, R, ... are any points.

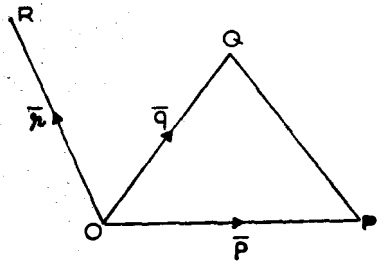


Fig. 1·11

Then the vectors $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$ .... are called the position vectors of P, Q, R, ... We generally denote the position vector of a point P as $\bar{p}$. Hence, $\overline{OP}=\bar{p}$, $\overline{OQ}=\bar{q}$, $\overline{OR}=\bar{r}$ etc. Now consider the fig. 1·11. We have $\overline{OP}+\overline{PQ}=\overline{OQ}$

$\therefore\ \overline{PQ}=\overline{OQ}-\overline{OP}\quad\therefore\ \overline{PQ}=\bar{q}-\bar{p}$.

Hence any vector can be written in terms of the position vectors of the extremities of the directed line segment. We shall use this result in many examples.

§1·21. The Section Formula :

(i) **Internal Division :** If P and Q are any two points having position vectors $\bar{p}$ and $\bar{q}$ with respect to O and R divides

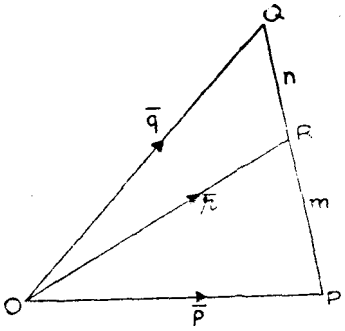


Fig. 1·12

seg PQ internally in the ratio $m:n$ then $\bar{r}=\dfrac{m\bar{q}+n\bar{p}}{m+n}$ where $\bar{r}$ is the positive vector of R.

Since $\dfrac{l(PR)}{l(RQ)}=\dfrac{m}{n}$

$\therefore\ n\cdot l(PR)=m\cdot l(RQ)$.

But the direction of $\overline{PR}$ and $\overline{RQ}$ is the same.

$\therefore\ n\cdot\overline{PR}=m\cdot\overline{RQ}$

$\therefore\ n(\bar{r}-\bar{p})=m(\bar{q}-\bar{r})$

$\therefore\ n\bar{r}-n\bar{p}=m\bar{q}-m\bar{r}$ $\quad\therefore\ m\bar{r}+n\bar{r}=m\bar{q}+n\bar{p}$

$\therefore\ (m+n)\bar{r}=m\bar{q}+n\bar{p}$ $\quad\therefore\ \bar{r}=\dfrac{m\bar{q}+n\bar{p}}{m+n}$ as required.

(ii) **External Division :**

If P and Q are any two points having position vectors $\bar{p}$ and $\bar{q}$ with respect to O and R divides PQ *externally* in the ratio $m:n$ then $\bar{r}=\dfrac{m\bar{q}-n\bar{p}}{m-n}$ where $\bar{r}$ is the position vector of R.

Since $\dfrac{l(\mathrm{PR})}{l(\mathrm{RQ})}=\dfrac{m}{n}$ in magnitude

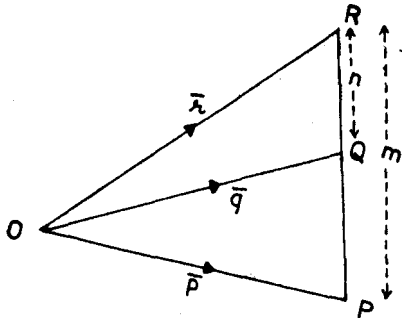


Fig. 1·13

$\therefore\ n\cdot l(\mathrm{PR})=m\cdot l(\mathrm{RQ})$ in magnitude.

But the directions of $\overline{\mathrm{PR}}$ and $\overline{\mathrm{RQ}}$ are opposite

$\therefore\ n\cdot\overline{\mathrm{PR}}=-m\cdot\overline{\mathrm{RQ}}$

$\therefore\ n(\bar{r}-\bar{p})=-m(\bar{q}-\bar{r})$

$\therefore\quad n\bar{r}-n\bar{p}=-m\bar{q}+m\bar{r}$

$\therefore\quad m\bar{q}-n\bar{p}=m\bar{r}-n\bar{r}$

$=(m-n)\bar{r}$

$\therefore\ \bar{r}=\dfrac{m\bar{q}-n\bar{p}}{m-n}$ as required.

These two formulae are of immense use in solving examples.

§ 1·22. Examples :

Ex. 1. If ABC is a triangle and G is its centroid, then prove that $\overline{\mathrm{GA}}+\overline{\mathrm{GB}}+\overline{\mathrm{GC}}=0$.

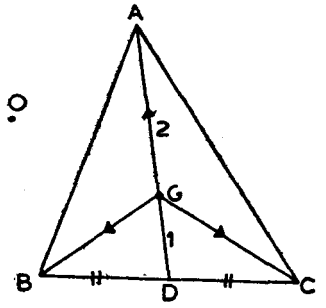


Fig. 1·14

Ans. Let D be the midpoint of seg BC. If O is any point in the plane and the position vectors of A, B, C with respect to O are $\bar{a}$, $\bar{b}$, $\bar{c}$ then by the section formula, $\bar{d}=\dfrac{\bar{b}+\bar{c}}{2}$, since D s the mid point of seg BC.

Also since $\dfrac{l(\mathrm{AG})}{l(\mathrm{GD})}=\dfrac{2}{1}$

$\therefore\ \bar{g}=\dfrac{2\bar{d}+\bar{a}}{2+1}$

$$\therefore \quad \bar{g} = \frac{2\left(\frac{\bar{b}+\bar{c}}{2}\right)+\bar{a}}{3} = \frac{\bar{a}+\bar{b}+\bar{c}}{3}$$

Now L. H. S. $= \overline{GA} + \overline{GB} + \overline{GC}$

$$= \bar{a} - \bar{g} + \bar{b} - \bar{g} + \bar{c} - \bar{g}$$
$$= \bar{a} + \bar{b} + \bar{c} - 3\bar{g}$$
$$= \bar{a} + \bar{b} + \bar{c} - 3\left(\frac{\bar{a}+\bar{b}+\bar{c}}{3}\right)$$
$$= \bar{a} + \bar{b} + \bar{c} - (\bar{a} + \bar{b} + \bar{c}) = 0 = \text{R. H. S.}$$

Ex. 2. If l, m, n are numbers, not all zero, such that $l+m+n=0$ and $l\bar{a}+m\bar{b}+n\bar{c}=0$ where $\bar{a}$, $\bar{b}$, $\bar{c}$ are the position vectors of A, B, C; then prove that A, B, C are collinear.

Ans. Suppose $l \neq 0$. $\because \; l = -m-n$ and

$$l \cdot \bar{a} + m \cdot \bar{b} + n \cdot \bar{c} = 0$$

$$\therefore \quad (-m-n)\,\bar{a} + m\bar{b} + n\bar{c} = 0$$

$$\therefore \quad -(m+n)\,\bar{a} + m\bar{b} + n\bar{c} = 0$$

$$\therefore \quad m\bar{b} + n\bar{c} = (m+n)\,\bar{a}$$

$$\therefore \quad \bar{a} = \frac{m\bar{b}+n\bar{c}}{m+n}. \quad \text{Since } l \neq 0 \quad \therefore \; (m+n) \neq 0$$

But by section formula, this means A divides seg BC in the ratio $n : m$, i. e. A, B, C are collinear.

Note : The results proved in Ex. 1 and 2 are very important and may be used in examples.

Ex. 3. If ABC is a triangle whose circumcentre is O and orthocentre is H, then prove that $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$.

Ans. Take position vectors with respect to O. Now we have to prove that $\bar{a} + \bar{b} + \bar{c} = \bar{h}$.

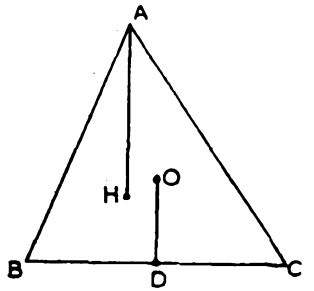


Fig. 1·15

If D is the mid point of BC then we know from geometry that

$$l(\text{AH}) = 2 \cdot l(\text{OD}) \text{ and AH} \parallel \text{OD}.$$

$$\therefore \quad \overline{AH} = 2\,\overline{OD} = 2\,\bar{d}$$

$$\text{But } \bar{d} = \frac{\bar{b}+\bar{c}}{2}$$

$$\therefore \quad \overline{AH} = 2\left(\frac{\bar{b}+\bar{c}}{2}\right)$$

$$\therefore \quad \overline{AH} = \bar{b} + \bar{c} \quad \therefore \quad \bar{h} - \bar{a} = \bar{b} + \bar{c}$$

$$\therefore \quad \bar{h} = \bar{a} + \bar{b} + \bar{c} \text{ as required.}$$

Exercise 1·2 :

(1) If G and G′ are the centroides of triangles ABC and A′B′C respectively then prove that $\overline{AA'}+\overline{BB'}+\overline{CC'}=3\overline{GG'}$.

2) If H is the orthocentre and O is the circumcentre of triangle ABC prove that $\overline{AH}+\overline{BH}+\overline{CH}=2\overline{OH}$.

3) If points P, Q, R, S have position vectors $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ such that $\bar{p}-\bar{q}=2(\bar{s}-\bar{r})$, show that the line QS and PR trisect each other.

(4) D, E, F are the mid points of the sides BC, CA, AB respectively of a triangle ABC. Show that $\overline{BC}=2\overline{EF}$ and $\overline{AD}+\overline{BE}+\overline{CF}=0$.

(5) Show that if P, Q, R, S are the mid points of the sides of a quadrilateral, then PQRS is a parallelogram.

(Hint : Show that $\overline{PQ}=\overline{SR}$).

(6) Show that the internal bisectors of the angles of a triangle are concurrent.

(7) If ABCD is a quadrilateral and P, Q are the mid points of AC and BD, show that $\overline{AB}+\overline{AD}+\overline{CB}+\overline{CD}=4\overline{PQ}$.

CHAPTER 2

Coordinate Systems, Distance and Section Formulae

The Coordinate Geometry provides a bridge between Geometry and Algebra and helps us to solve geometrical problems algebraically or analytically. Consider a line-segement of length $\sqrt{2}$. Even with the most accurate measurement, we shall not be able to measure this segment to get the length $\sqrt{2}$. Similar is the case of finding the angle between two lines. In all such cases, the Coordinate Geometry enables us to find them with utmost accuracy and exactness of numbers.

The association between Algebra and Geometry is made possible by assigning numbers to points. You already know how to set up coordinate system on a line by selecting a point O, called origin, on the line and assigning positive real numbers to the points on the right of O and negative real numbers to the points on the left of O. Thus, there is one-one correspondence between the points on the line and the real numbers. A number which is thus associated with a point on the line is called the *coordinate* of that point.

You know how to set up a coordinate system in a plane by taking a horizonial line called x axis and a vertical line called the y axis. Let us revise this.

§ 2·1. Rectangular Cartesian Coordinate System :

Look at fig. 2·1. We have drawn two mutually perpendicular lines X'OX and Y'OY in the plane of the paper. These lines are called the coordinate axes. Specifically, the line X'OX is called the x axis and the line Y'OY is called the y axis. O is called the origin. As you can see, these two lines divide the plane of the paper into four disjoint convex sets. These sets are called *quadrants*.

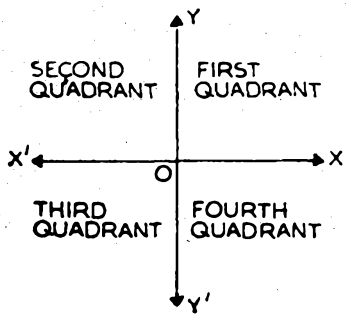


Fig. 2·1

Specifically the upper right quadrant is called the *first quadrant*; the upper left quadrant is called the *second quadrant*; the lower left quadrant is called the *third quadrant*; the lower right quadrant is called the *fourth quadrant* as shown in fig. 2·1.

Using the system of coordinate axes, we see that every point on the plane is either on the x axis or on the y axis or is in one of the quadrants. What we desire is the exact location of a point to be written in terms of an ordered pair of real numbers. This is done as follows :

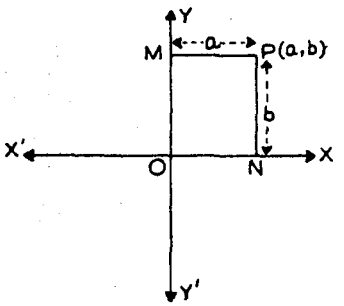


Fig. 2·2

Let P be any point in the plane of the paper in which the coordinate axes X'OX and Y'OY have been drawn. We draw PM perpendicular to the y axis and PN perpendicular to the x axis as shown in fig. 2·2.

The distances PM and PN can now be measured. We define the following :

Distance of P from y axis $= d(P, M)$
$= x$ Coordinate of P. [Also called "abscissa" of P]
Distance of P from x axis $= d(P, N)$
$= y$ coordinate of P. [Also called "ordinate" of P]

Thus, the coordinates of the point P are nothing but the distances of the point P from the two axes of coordinates.

Sign Convention : Since $d(P, M)$ is measured along the x-axis and $d(P, N)$ is measured along the y axis, we define the following sign convention :

"All distances measured along the x axis to the right of O are positive and those measured to the left of O are negative. All distances measured along the y axis upwards from O are positive and those measured downwards from O are negative. (Note : The x-axis is always drawn in the horizontal direction and the y axis in the vertical direction).

Hence the coordinates of a point may be positive or negative depending on the position of the point in the various quardrants.

Notation : If, in fig. 2·2, $l(PM) = a$ and $l(PN) = b$, then we denote the point P as $P \equiv (a, b)$. Thus, the coordinates of a

point form an ordered pair with the x coordinate written as the first member of this ordered pair.

§ 2·11. Special Points : Consider fig. 2·3.

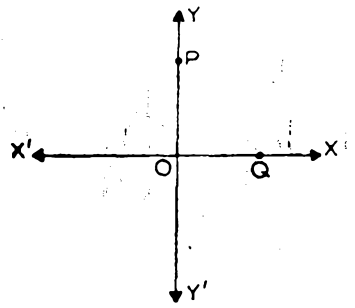


Fig. 2·3

(i) We have taken a point P on the y axis. Obviously, the distance of P from y axis is then zero and hence the x coordinate of P is zero.

This applies to any point on the y axis and hence we have the result that :

" Every point on the y axis has x coordinate $= 0$."

(ii) It is clear in a similar way that the point Q, which is on the x axis, has y coordinate $= 0$.

This is true for any point on x axis and hence we have :

" Every point on the x axis has y coordinate $= 0$. "

(iii) Since O lies on x axis as well as y axis, the coordinates of O are (0, 0).

[**Note :** In Fig. 2·3, if $l(QO) = a$ then $Q \equiv (a, 0)$ and if $l(PO) = b$, then $P \equiv (0, b)$]

§ 2·12. Examples :

Ex. 1. A (-2, 0) and B (0, 5) are two points pnd O is the origin. Find the area of $\triangle$ OAB.

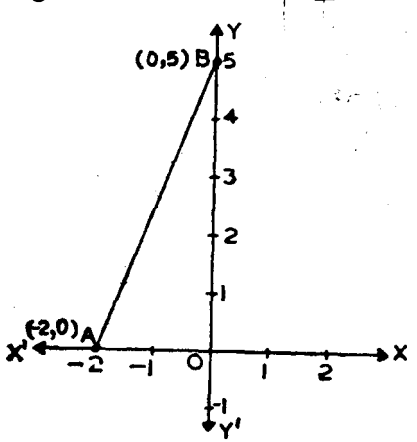


Fig. 2·4

Ans. See fig. 2·4. Since A has ordinate $= 0$, it lies on x axis.

Similarly, B is on y axis as shown.

$\therefore$ $\triangle$ OAB is right angled at O.

Also $l(OA) = 2$ and $l(OB) = 5$.

$\therefore$ Area of $\triangle$ OAB.

$= \frac{1}{2} \cdot l(OA) \cdot l(OB)$

$= \frac{1}{2}(2)(5)$

$= 5$ square units.

Ex. 2. A $(-3, 0)$ and B $(2, 0)$ are two points. Find a point C such that $\triangle$ ABC is right angled at B with $l(BC) = 3$.

Ans. See fig. 2·5.

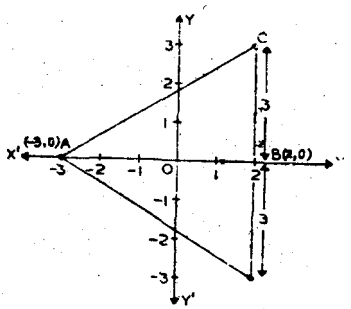


Fig. 2·5

$\because$ A and B have ordinates $= 0$
$\therefore$ they lie on x-axis such that $l(OA) = 3$ units to the left of O and $l(OB) = 2$ units. Draw a perpendiaular to the x-axis at B. The point C is on this perpendicular but can be either above the x-axis or below the x-axis. $\therefore$ C has two positions as shown in fig. 2·5. In both cases $l(BC) = 3$ units. $\therefore$ x coordinate of $C = l(OB) = 2$ and y coordinate of $C = l(CB) = 3$ when C is in the first quadrant and -3 when C is in the fourth quadrant.

$\therefore$ $C \equiv (2, 3)$ or $C \equiv (2, -3)$.

Exercise 2·1 :

(1) P is a point and PM and PN are drawn perpendicular from P on the x and y axes meeting them at M and N respectively. Solve the following :—

(a) If $P \equiv (3, 5)$, find the coordinates of M and N.

(b) If $P \equiv (-3, 4)$, find the coordinates of M and N.

(c) If $M \equiv (3, 0)$ and $N \equiv (0, -5)$, find coordinates of P.

(d) If $P \equiv (3, 4)$, find $l(OM)$, $l(MP)$ and hence find $l(OP)$.

(2) A (2, 0), B (4, 0) and C (4, 5) are three points. Find the area of $\triangle$ABC.

(3) A (0, 2), B (0, 5) and C $(-6, 5)$ are three points. Find the area of $\triangle$ABC.

(4) A (0, 5) and B (0, 9) are two points. Find a point C such that $\triangle$ABC is right angled at A with $l(AC) = 2$ units.

§ 2·13. Position vector as an ordered pair :

Consider the rectangular axes X'OX and Y'OY. Let P (x, y) be any point in the plane. Let position vector of P with respect to O be $\overline{OP} = \bar{p}$.

Now suppose $\bar{i}$ denotes the unit vector along OX and $\bar{j}$ denotes the unit vector along OY. Draw PM perpendicular to OX meeting it at M. Since $l(OM) = x$.

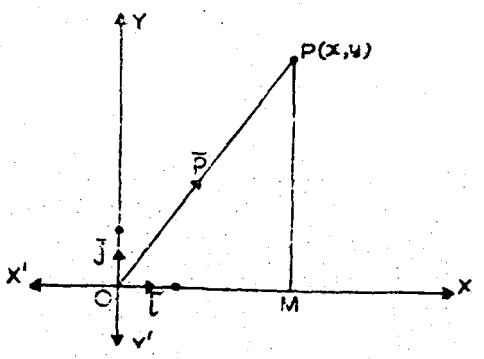


Fig. 2·6

$\therefore \overrightarrow{OM} = x \cdot \vec{i}$.

Also $l(MP) = y$

$\therefore \overrightarrow{MP} = y \cdot \vec{j}$,

Now $\vec{p} = \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = x \cdot \vec{i} + y \cdot \vec{j}$.

Hence if $P \equiv (x, y)$, we have

$$\vec{p} = x \cdot \vec{i} + y \cdot \vec{j}.$$

Since $\vec{i}$ and $\vec{j}$ are fixed vectors, $\vec{p}$ is determined if x and y are given. Hence we can write $\vec{p} \equiv (x, y)$.

In this way, a position vector in a plane can be written as an ordered pair.

This ordered pair may be written as (x, y) or as $\begin{pmatrix} x \\ y \end{pmatrix}$.

§ 2·14. Polar coordinates :

This is another system for representing points in a plane. Consider a plane. Let OX be any line in this plane. This line is called the *initial line.* Consider the point O on it, to be called the *pole.*

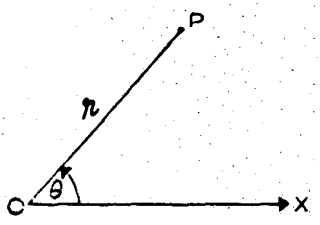


Fig. 2·7

Then if P is any point in the plane, we denote the distance OP by r and the $m\angle XOP$ by θ. Then r and θ are called the polar coordinates of the point P which is written as $P(r, \theta)$.

We take r to be always positive and θ is the measure of the rotation from OX to OP.

§ 2·15. Conversion from cartesian to polar coordinates :

Consider the system of rectangular axes X'OX and Y'OY. Take OX as the initial line and O as the pole. Then if

$m\angle XOP = \theta$ and $l(OP) = r$, we have

$\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ by the definition of trigonometric ratios.

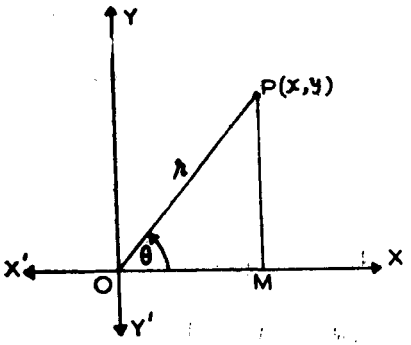


Fig. 2·8

$\therefore\ x = r\cos\theta$ and $y = r\sin\theta$. These are the formulae to convert x, y in terms of r and θ and conversely. e. g, if a point has $\left(2, \frac{\pi}{4}\right)$ as its polar coordinates then its cartesian coordinates are

$$\left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right)$$
$$= \left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$$
$$= (\sqrt{2}, \sqrt{2}).$$

In this way you can fill in the blanks in the following:

	Polar Coordinates	Cartesian Coordinates
(1)	$\left(3, \frac{\pi}{6}\right)$	$(-, -)$
(2)	$\left(4, \frac{\pi}{3}\right)$	$(-, -)$
(3)	$\left(2, \frac{2\pi}{3}\right)$	$(-, -)$
(4)	$\left(-, \frac{\pi}{6}\right)$	$(-, 2)$
(5)	$\left(-, \frac{\pi}{3}\right)$	$(3, -)$

§ 2·2. Distance formula :

We shall now establish a formula to find the distance between two points A and B if the coordinates of A and B are given.

Notation : Henceforth we shall use the letters x and y for denoting the x coordinate and y coordinate of a point. If more than one point is used, the coordinates can be distinguished by assigning suffixes to x and y, e. g. (x_1, y_1); (x_2, y_2); (x_3, y_3) etc

Theorem 2.1: If A (x_1, y_1) and B (x_2, y_2) are two points then

$$l(AB) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof :

Draw the coordinate axes. Let A (x_1, y_1) and B (x_2, y_2) be. the given points. Draw AA′ and BB′ perpendicular to x axis

meeting it at A′ and B′ respectively. Draw AR perpendicular to BB′ meeting BB′ at R.

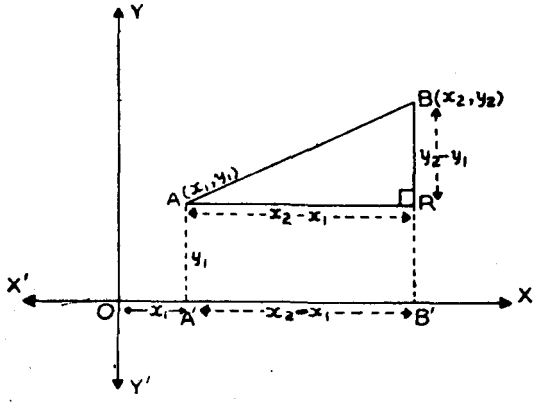


Fig. 2·9

∴ If O is the origin,

$l(OA') = x_1$; $l(AA') = y_1$; $l(OB') = x_2$; $l(BB') = y_2$.

Since △ ABR is right angled at R, we have

$$[l(AB)]^2 = [l(AR)]^2 + [l(RB)]^2 \quad .. \quad (i)$$

As ARB′A′ is a rectangle

∴ $l(AR) = l(A'B') = l(OB') - l(OA') = x_2 - x_1$

and $l(BR) = l(BB') - l(RB') = l(BB') - l(AA') = y_2 - y_1$.

∴ By (i), $[l(AB)]^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

∴ $l(AB) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ as required.

Note: Since $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, the order in which the coordinates are subtracted is immaterial.

§ 2·21. Examples:

Ex. 1. Find $d(A, B)$ if $A \equiv (7, 4)$ and $B \equiv (3, 1)$.

Ans. $l(AB) = \sqrt{(7-3)^2 + (4-1)^2} = \sqrt{4^2 + 3^2} = \sqrt{25}$

$= 5$ units.

Ex. 2. Find $d(P, Q)$ if $P \equiv (-1, 3)$ and $Q \equiv (-6, -9)$.

Ans. $l(PQ) = \sqrt{[(-1) - (-6)]^2 + [3 - (-9)]^2}$

$= \sqrt{(5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169}$

$= 13$ units.

Ex. 3. If A(6, 1); B.(−1, 8) and C(3, −2) form a △ ABC; show that △ ABC is right angled.

Ans. We shall find $l(AB)$, $l(BC)$ and $l(AC)$.

$$l(AB)=\sqrt{[6-(-1)]^2+[1-8]^2}$$
$$=\sqrt{(7)^2+(-7)^2}=\sqrt{49+49}=\sqrt{98}.$$
$$l(BC)=\sqrt{(-1-3)^2+[8-(-2)]^2}$$
$$=\sqrt{(4)^2+(10)^2}=\sqrt{16+100}=\sqrt{116}$$
$$l(AC)=\sqrt{(6-3)^2+[1-(-2)^2]}$$
$$=\sqrt{(3)^2+(3)^2}=\sqrt{9+9}=\sqrt{18}$$

$\therefore$ We can see that

$$[l(AB)]^2+[l(AC)]^2=98+18=116=[l(BC)]^2$$

$\therefore$ △ABC is right angled at A.

Ex. 4. Find the coordinates of the circumcentre and the length of the circumradius of △ ABC if A ≡ (4, 1), B ≡ (−3, −6) and C ≡ (−2, 1)

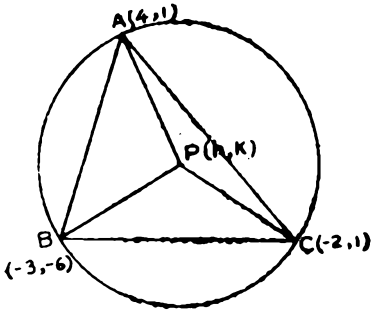


Fig. 2.10

Ans. The circumcentre of △ ABC is the centre of the circle passing through the points A, B, C. Hence if P is the circumcentre of △ ABC, then

$$l(PA)=l(PB)=l(PC)$$

[See fig. 2·10

Let P be (h, k). $\because$ $l(PA)=l(PB)$

$$\therefore \sqrt{(h-4)^2+(k-1)^2}=\sqrt{(h+3)^2+(k+6)^2}$$
$$\therefore (h-4)^2+(k-1)^2=(h+3)^2+(k+6)^2$$
$$\therefore h^2-8h+16+k^2-2k+1=h^2+6h+9+k^2+12k+36.$$
$$\therefore -8h-2k+17=6h+12k+45$$
$$\therefore -14h-14k-28=0$$
$$\therefore h+k+2=0 \quad \ldots\ \ldots\ (i)$$
$$\because l(PB)=l(PC)$$
$$\therefore \sqrt{(h+3)^2+(k+6)^2}=\sqrt{(h+2)^2+(k-1)^2}$$
$$\therefore (h+3)^2+(k+6)^2=(h+2)^2+(k-1)^2$$

$\therefore \quad h^2+6h+9+k^2+12k+36=h^2+4h+4+k^2-2k+1$

$\therefore \quad 6h+12k+45=4h-2k+5$

$\therefore \quad 2h+14k+40=0$

$\therefore \quad h+7k+20=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$

Solving (i) and (ii) simultaneously,
we get $h=1$ and $k=-3$

$\therefore$ Circumcentre $P\equiv(1, -3)$.

Also circumradius $= l(PA)=\sqrt{(1-4)^2+(-3-1)^2}$
$=\sqrt{9+16}=5$ units.

Ex. 5. Find y if the distance between $(2, -3)$ and $(-1, y)$ is 5 units.

Ans. Let $A\equiv(2, -3)$; $B\equiv(-1, y)$

Since $l(AB)=5 \quad \therefore \quad \sqrt{[2-(-1)]^2+[-3-y]^2}=5$

$\therefore \quad (3)^2+(y+3)^2=25 \quad \therefore \quad (y+3)^2=25-9$

$\therefore \quad (y+3)^2=16 \quad \therefore \quad y+3=\pm 4$ (Note this step).

$\therefore \quad y=-3\pm 4 \quad \therefore \quad y=1$ or -7.

Ex. 6. Find a point on the x axis whose distance from (7, 5) is 13 units.

Ans. Let $A\equiv(7, 5)$. Let P be the required point on x axis such that $l(AP)=13$.

Since P lies on x axis, y coordinate of $P=0$

$\therefore$ Let $P\equiv(h, 0)$.

Since $l(AP)=13 \quad \therefore \quad \sqrt{(h-7)^2+(0-5)^2}=13$

$\therefore \quad (h-7)^2+25=169 \quad \therefore \quad (h-7)^2=169-25$

$\therefore \quad (h-7)^2=144 \quad \therefore \quad h-7=\pm 12 \quad \therefore \quad h=7\pm 12$

$\therefore \quad h=19$ or -5.

$\therefore$ Required point P is $(19, 0)$ or $(-5, 0)$.

Exercise 2.2 :

(1) Find the distance between the following pairs of points :—
(a) (3, −1) and (7, −4); (b) (2, −5) and (0, 2);
(c) (6, −8) and (0, 0).

(2) If A is (1, −2) and B is (4, y), find the possible values of y so that $l(AB)=5$.

(3) If the point $(9, a)$ is at a distance of 5 units from the point $(a, 2)$ find a.

(4) Find the coordinates of the points on x axis which are at distance 10 units from $(3, -6)$.

(5) Find the coordinates of the points on y axis which are at distance of 13 units from $(5, 4)$.

(6) Find a, if the triangle formed by A $(4, 3)$; B $(6, -2)$ and C $(a, -3)$. is right angled at A.

(7) Prove that $(-1, 2)$, $(3, -1)$ and $(2, 6)$ are the vertices of an isosceles right angled triangle.

(8) A is the vertex and BC is the base of an isosceles triangle ABC If $A \equiv (2, a)$, $B \equiv (5, 4)$ and $C \equiv (1, 0)$, find the value of a.

(9) Show that the points $(1, 2)$; $(1, 6)$ and $(\sqrt{12}+1, 4)$ are the vertices of an equilateral triangle.

(10) Find the coordinates of the circumcentre and length of the circumradius of the triangle whose vertices are :

(a) $(0, 0)$; $(6, 0)$; $(0, 10)$.
(b) $(4, 6)$; $(-2, -2)$; $(-3, -1)$.
(c) $(3, -2)$; $(4, 3)$; $(-6, 5)$.
(d) $(3, -9)$; $(0, 0)$; $(-5, -5)$.

(11) If $(3, -1)$ is the circumcentrle of the triangle whose vertices are $(0, 3)$; $(-1, b)$ and $(a, -1)$, find a and b.

(12) If the distance between (na, nb) and (a, b) is three times the distance between $(4a, 4b)$ and (a, b), find n.

§ 2·3. Division of a line segment :

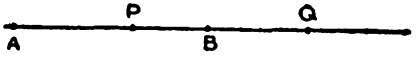


Fig. 2·11

In fig. 2·11, AB is a line segment and we have taken a point P on it between A and B and another point Q on it but outside seg AB.

We say that P divides seg AB internally in the ratio $\frac{l(AP)}{l(PB)}$ and Q divides seg AB externally in their ratio $\frac{l(AQ)}{l(QB)}$.

If we are given the coordinates of the points A and B and the ratio in which a point divides seg AB; then we can find the coordinates of the point of division. The formula for this is called the section formula.

§ 2·31. Section formula (internal division) :

Theorem 2·2 : If $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ and P divides seg

AB ***internally*** in the ratio $m : n$, then the coordinates of P are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right).$$

Proof:

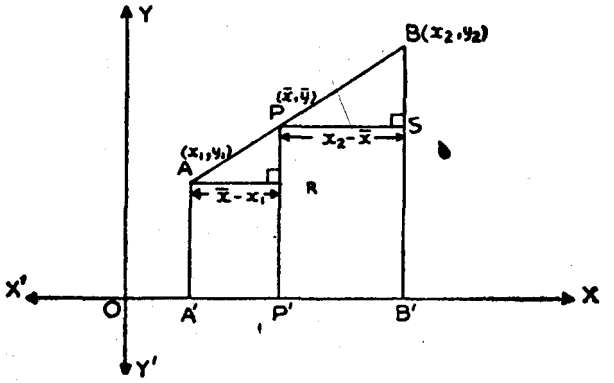


Fig. 2·12

Draw the coordinates axes. A and B are the given points. P lies between A and B such that $\frac{l(AP)}{l(PB)} = \frac{m}{n}$. Let $P \equiv (\bar{x}, \bar{y})$.

Draw AA', BB' and PP' perpendicular to OX and AR perpendicular to PP'; PS perpendicular to BB' as shown in figure 2·12.

$\therefore$ $l(OA') = x_1$; $l(AA') = y_1$; $l(OB') = x_2$; $l(BB') = y_2$;
$l(OP') = \bar{x}$; $l(PP') = \bar{y}$.

Since the triangles APR and PBS are similar

$$\therefore \quad \frac{l(AR)}{l(PS)} = \frac{l(PR)}{l(BS)} = \frac{l(AP)}{l(PB)}$$

$$\therefore \quad \frac{l(AR)}{l(PS)} = \frac{l(PR)}{l(BS)} = \frac{m}{n}$$

$$\text{Now consider } \frac{l(AR)}{l(PS)} = \frac{m}{n} \qquad \ldots\ldots (1)$$

Since ARP'A' is a rectangle

$\therefore$ $l(AR) = l(A'P')$
$= l(OP') - l(OA') = \bar{x} - x_1$

Similarly PSB'P' is a rectangle

$\therefore$ $l(PS) = l(P'B') = l(OB') - l(OP') = x_2 - \bar{x}$

$\therefore$ (i) gives $\dfrac{\bar{x}-x_1}{x_2-\bar{x}}=\dfrac{m}{n}$

$\therefore \quad n\bar{x}-nx_1=mx_2-m\bar{x}$

$\therefore \quad m\bar{x}+n\bar{x}=mx_2+nx_1$

$\therefore \quad \bar{x}(m+n)=mx_2+nx_1$

$\therefore \quad \bar{x}=\dfrac{mx_2+nx_1}{m+n}$

Similarly $\dfrac{l(\mathrm{PR})}{l(\mathrm{BS})}=\dfrac{m}{n}$ will give us $\bar{y}=\dfrac{my_2+ny_1}{m+n}$

$\therefore \quad \mathrm{P}\equiv\left(\dfrac{mx_2+nx_1}{m+n},\ \dfrac{my_2+ny_1}{m+n}\right)$

Note : While using this formula, it is important to note the positions of m and n and understand how to link up the numbers m and n with the coordinates of A and B. This will be clear from the following figure :

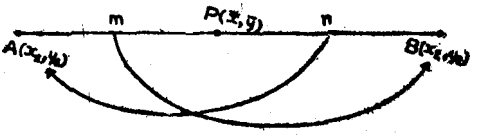


Fig. 2·13

The number m, associated with seg AP, will be multiplied to the coordinates of Q and similarly, n will be multiplied to the coordinates of P. $\therefore$ We get

$$\bar{x}=\frac{mx_2+nx_1}{m+n} \quad \text{and} \quad \bar{y}=\frac{my_2+ny_1}{m+n}.$$

§ 2·32. Example :

Ex. 1. If A $\equiv$ (2, -3) and B $\equiv$ (5, 6) and P divides seg AB internally in the ratio 2 : 1, find the coordinates of P.

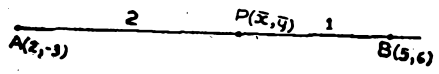


Fig. 2·14

Ans. Let P $\equiv (\bar{x}, \bar{y})$.

By section formula

$$\bar{x} = \frac{2(5)+1(2)}{2+1} = \frac{10+2}{3} = 4$$

$$\bar{y} = \frac{2(6)+1(-3)}{2+1} = \frac{12-3}{3} = 3$$

$\therefore$ P $\equiv (4, 3)$.

Ex. 2. P divides seg AB internally in the ratio 3 : 2. If A $\equiv (1, -2)$ and P $\equiv (4, 7)$, find the coordinates of B.

Ans. Let B $\equiv (h, k)$.

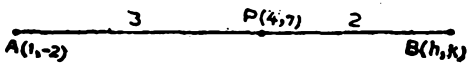


Fig. 2·15

$\therefore$ By section formula

$$\frac{3(h)+2(1)}{3+2} = 4 \quad \therefore \quad 3h+2 = 20 \quad \therefore \quad h = 6$$

and $$\frac{3(k)+2(-2)}{3+2} = 7 \quad \therefore \quad 3k-4 = 35 \quad \therefore \quad k = 13$$

$\therefore$ B $\equiv (6, 13)$.

Ex. 3. A $\equiv (-2, 2)$ and B $\equiv (3, 7)$ are two points. A point P with abscissa 2 lies on AB. Find the ratio in which P divides AB and the ordinate of P.

Ans. Let $\frac{l(\text{AP})}{l(\text{PB})} = \frac{m}{n}$ and P $\equiv (2, k)$

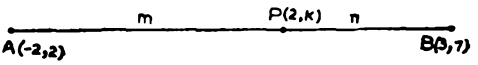


Fig. 2·16

$\therefore$ By section formula :

$$\frac{m(3)+n(-2)}{m+n} = 2$$

$\therefore$ $3m-2n = 2m+2n \quad \therefore \quad m = 4n \quad \therefore \quad \frac{m}{n} = \frac{4}{1}$

$\therefore$ P divides seg AB internally in the ratio 4 : 1. Also using $\frac{m}{n} = \frac{4}{1}$.

$$k = \frac{4(7)+1(2)}{4+1} = \frac{28-2}{5} = 6.$$

$\therefore$ Ordinate of P = 6.

§ 2·33. Corollaries from the section formula :

Cor. 1. $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ are two points then C, the mid-point of seg AB is given by

$$C \equiv \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Can you check this ?

Cor. 2. Centroid of a triangle: Let $A(x_1, y_1)$; $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC, then the centroid G of triangle ABC has coordinates:

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Proof:

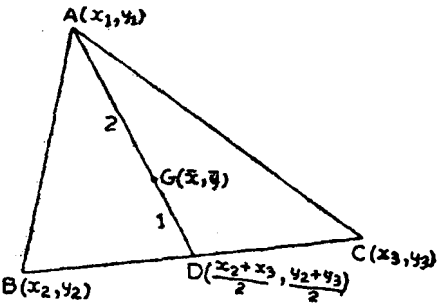


Fig. 2·17

Let D be the mid-point of seg BC. Then

$$D \equiv \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

Let $G(\bar{x}, \bar{y})$ be the centroid of $\triangle ABC$.

$\therefore$ G divides seg AD internally in the ratio 2:1.

$\therefore$ By section formula

$$\bar{x} = \frac{2\left(\frac{x_2+x_3}{2}\right)+1(x_1)}{2+1} = \frac{x_1+x_2+x_3}{3}$$

and $\bar{y} = \frac{2\left(\frac{y_2+y_3}{2}\right)+1(y_1)}{2+1} = \frac{y_1+y_2+y_3}{3}$ as required.

Cor. 3. Conditions for a parallelogram: Let ABCD be a parallelogram where $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$ and $D \equiv (x_4, y_4)$.

Then $x_1+x_3 = x_2+x_4$ and $y_1+y_3 = y_2+y_4$.

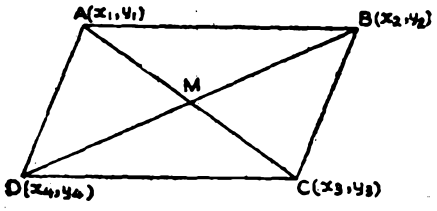


Fig. 2·18

Proof :

Since ABCD is a parallelogram,

$\therefore$ Seg AC and seg BD bisect each other at M.

Since M is mid-point of seg AC

$$\therefore \quad M \equiv \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right) \qquad \ldots\ldots \text{(i)}$$

Since M is mid-point of seg BD

$$\therefore \quad M \equiv \left(\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}\right) \qquad \ldots\ldots \text{(ii)}$$

By (i) and (ii) $\frac{x_1+x_3}{2} = \frac{x_2+x_4}{2}$ and $\frac{y_1+y_3}{2} = \frac{y_2+y_4}{2}$

$\therefore \quad x_1+x_3 = x_2+x_4$ and $y_1+y_3 = y_2+y_4$.

§ 2·34. Examples :

Ex. 1. Seg AB is the diameter of a circle with centre C. If $A \equiv (1, -2)$ and $C \equiv (-3, 4)$, find the coordinates of B.

Ans. Let $B \equiv (h, k)$. Since C $(-3, 4)$ is the mid-point of seg AB, we get

$$\frac{1+h}{2} = -3 \text{ and } \frac{(-2)+k}{2} = 4$$

$$\therefore \quad h = -7 \quad \text{and} \quad k = 10$$

$$\therefore \quad B \equiv (-7, 10).$$

Ex. 2. ABC is a triangle whose centroid is G. If $A \equiv (-3, 1)$; $B \equiv (2, b)$; $C \equiv (a, -4)$ and $G \equiv (1, -1)$; find a and b.

Ans. By the formula for centroid :

$$\frac{(-3)+2+a}{3} = 1 \qquad \therefore \quad a = 4$$

$$\text{and} \quad \frac{1+b+(-4)}{3} = -1 \qquad \therefore \quad b = 0.$$

Ex. 3. ABCD is a parallelogram. If $A \equiv (-2, -3)$, $B \equiv (2, 7)$; $C \equiv (1, 5)$, find the coordinates of D.

Ans. Let $D \equiv (h, k)$.

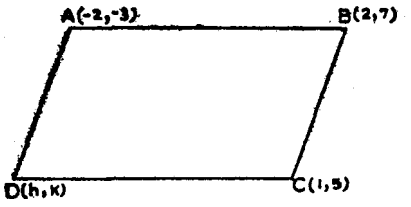


Fig. 2·19

$\therefore \quad h+2=1+(-2)$

$k+7=5+(-3)$

$\therefore \quad h=-3$

$\therefore \quad k=-5$

$\therefore \quad D \equiv (-3,-5).$

Ex. 4. If $(-2,1)$; $(-1,5)$; $(3,2)$ are the midpoints of the sides of a triangle, find the coordinates of the vertices of the triangle.

Ans. Let A (x_1, y_1): B (x_2, y_2) and C (x_3, y_3) be the vertices of the triangle and D $(-2, 1)$: E $(-1, 5)$ and F $(3, 2)$, be the mid-points of BC, AB, CA respectively. Since AFDE is a parallelogram.

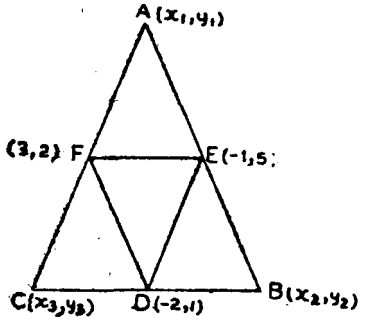


Fig. 2·20

$\therefore \quad x_1+(-2)=3+(-1)$

$\therefore \quad x_1=4$

and $\quad y_1+1=2+5$

$\therefore \quad y_1=6$

$\therefore \quad A \equiv (4, 6)$

In a similar way, we can prove

$B \equiv (-6, 4)$ and

$C \equiv (2, -2).$

Note: This example could also be asked as follows:

"If $(-2, 1)$; $(-1, 5)$ and $(3, 2)$ are three vertices of a parallelogram, find the fourth vertex."

It should be understood that, since the order of the points forming the parallelogram is not given, there exist 3 different parallelograms and not only one as in Ex. 3.

§ 2·35. Section formula : (External division) :

Theorem 2·3. If $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ and P divides seg AB externally in the ratio $m:n$, then the coordinates of P are $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$.

Proof :

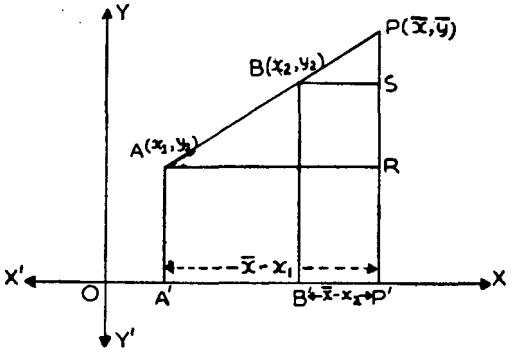


Fig. 2·21

Draw the coordinate axes. A and B are the given points. P divides seg AB externally such that $\frac{l(AP)}{l(PB)} = \frac{m}{n}$.

Let $P \equiv (\bar{x}, \bar{y})$.

Draw AA', BB' and PP' perpendicular to OX; AR, BS perpendicular to PP' as shown in the fig. 2·21.

$\therefore\ l(OA') = x_1,\ l(AA') = y_1\ ;\ l(OB') = x_2\ ;\ l(BB') = y_2\ ;$ $l(OP') = \bar{x},\ l(PP') = \bar{y}$.

Since the triangles APR and BPS are similar,

$$\therefore\ \frac{l(AR)}{l(BS)} = \frac{l(PR)}{l(PS)} = \frac{l(AP)}{l(PB)}$$

$$\therefore\ \frac{l(AR)}{l(BS)} = \frac{l(PR)}{l(PS)} = \frac{m}{n}$$

Now consider $\frac{l(AR)}{l(BS)} = \frac{m}{n}$ .. .. .. (i

Since ARP'A' is a rectangle,

$\therefore\ l(AR) = l(A'P') = l(OP') - l(OA') = \bar{x} - x_1$

Since BSP'B' is a rectangle,

$\therefore\ l(BS) = l(B'P') = l(OP') - l(OB') = \bar{x} - x_2$

$\therefore$ By (i) $\frac{\bar{x} - x_1}{\bar{x} - x_2} = \frac{m}{n}$

$$\therefore\ m\bar{x} - mx_2 = n\bar{x} - nx_1$$

$$\therefore\ m\bar{x} - n\bar{x} = mx_2 - nx_1$$

$$\therefore\ \bar{x}(m - n) = mx_2 - nx_1$$

$$\therefore\ \bar{x} = \frac{mx_2 - nx_1}{m - n}$$

Similarly by considering $\frac{l(PR)}{l(PS)} = \frac{m}{n}$,

we shall get $\bar{y} = \frac{my_2 - ny_1}{m-n}$.

$$\therefore P \equiv \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right)$$

Note: In this formula, we can write

$$P \equiv \left(\frac{mx_2 + (-n)x_1}{m+(-n)}, \frac{my_2 + (-n)y_1}{m+(-n)} \right)$$

which is the same as if the division is internal and the ratio is written as $m : (-n)$.

Hence we can treat an external division with positive ratio as an internal division with negative ratio and calculate the coordinates as we did in the internal division examples.

§ 2·36. Examples:

Ex. 1. *If* $A \equiv (-3, 1)$ and $B \equiv (4, 7)$ and P divides seg AB externally in the ratio 3 : 2, find the coordinates of P.

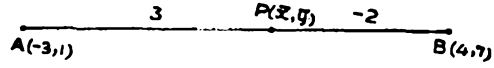


Fig. 2·22

Here we can take P to divide AB internally in the ratio $3 : -2$.

Let $P \equiv (\bar{x}, \bar{y})$

$$\therefore \bar{x} = \frac{3(4) + (-2)(-3)}{3+(-2)} = \frac{12+6}{1} = 18.$$

$$\bar{y} = \frac{3(7) + (-2)(1)}{3+(-2)} = \frac{21-2}{1} = 19.$$

$\therefore P \equiv (18, 19)$.

Ex. 2. If $A \equiv (3, -2)$ and $B \equiv (5, 4)$, find the ratio in which the y axis divides seg AB. Is the division internal or external? Find the point of intersection of y axis with AB.

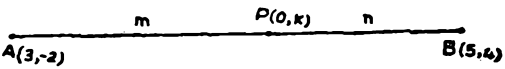


Fig. 2·23

Let the y axis intersect AB at P such that $\frac{l(AP)}{l(PB)} = \frac{m}{n}$.

Since P lies on y axis, x coordinate of P$=0$.

$\therefore$ Let P$\equiv(0, k)$.

Now since x coordinate of P$=0$, $\frac{m(5)+n(3)}{m+n} = 0$

$\therefore \quad 5m+3n=0 \quad \therefore \quad 5m=-3n \quad \therefore \quad \frac{m}{n} = -\frac{3}{5}$

Since the ratio is negative, we conclude that y axis divides seg AB externally in the ratio 3 : 5.

Using the ratio to be $-3:5$ in fig. 2·23, we get

$$k=\frac{(-3)(4)+(5)(-2)}{(-3)+5} = \frac{-12-10}{2} = -11.$$

$\therefore$ The y axis intersects seg AB at $(0, -11)$.

Ex. 3. A$\equiv(2, -1)$; B$\equiv(-1, -5)$. Seg AB is extended to P such that $l(AP)=3\cdot l(AB)$. Find the coordinates of P.

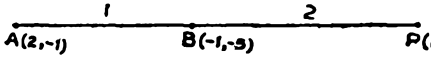


Fig. 2·24

Let P$\equiv(h, k)$.

Since B lies between A and P, we can consider B divides seg AP internally in the ratio $\frac{l(AB)}{l(BP)}$.

Now $\quad l(AP)=3\cdot l(AB)$

$\therefore \quad l(AB+BP)=3\cdot l(AB)$

$\therefore \quad l(AB)+l(BP)=3\cdot l(AB)$

$\therefore \quad 2[l(AB)]=l(BP)$

$\therefore \quad \frac{l(AB)}{l(BP)} = \frac{1}{2}$ giving the ratio.

Since x coordinte of B$=-1 \quad \therefore \quad \frac{1(h)+2(2)}{1+2} = -1$

$\therefore \quad h+4=-3 \quad \therefore \quad h=-7$

Also since y coordinate of B$=-5$

$\therefore \quad \frac{1(k)+2(-1)}{1+2} = -5$

$\therefore \quad k-2=-15 \quad \therefore \quad k=-13 \quad \therefore \quad P\equiv(-7, -13)$.

Ex. 4. If $A \equiv (-5, 8)$: $B \equiv (1, 5)$, find the points of trisection of seg AB.

Ans. The points of trisection of seg AB are the two points P and Q which divide seg AB into 3 equal parts.

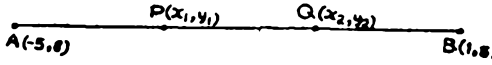


Fig. 2·25

$\therefore$ P divides seg AB internally in the ratio 1:2 and Q divides seg AB internally in the ratio 2:1. Let $P \equiv (x_1, y_1)$

$\therefore$ By section formula :

$$x_1 = \frac{1(1)+2(-5)}{1+2} = -3$$

$$y_2 = \frac{1(5)+2(8)}{1+2} = 7 \qquad \therefore \quad P \equiv (-3, 7).$$

Now we may similarly find Q as it divides seg AB internally in the ratio 2 : 1 or alternatively we may consider Q to be the midpoint of seg PB and get

$$Q \equiv \left(\frac{-3+1}{2}, \frac{7+5}{2}\right) \equiv (-1, 6).$$

Exercise 2·3 :

(1) Find the coordinates of the point P which divides seg AB internally in the ratio $m : n$ in the following cases :—

(i) $A \equiv (-3, -4)$; $B \equiv (2, 5)$; $m : n = 2 : 1$

(ii) $A \equiv (2, -5)$; $B \equiv (-3, -7)$; $m : n = 3 : 5$

(iii) $A \equiv (-1, -1)$; $B \equiv (7, 4)$; $m : n = 2 : 3$.

(2) Find the coordinates of the point P which divides seg AB externally in the ratio $m : n$ in the following cases :

(i) $A \equiv (3, 4)$; $B \equiv (1, -4)$; $m : n = 2 : 1$

(ii) $A \equiv (2, 4)$; $B \equiv (-3, -5)$; $m : n = 3 : 5$

(iii) $A \equiv (-5, -3)$; $B \equiv (-4, 7)$; $m : n = 4 : 3$.

(3) $A \equiv (-2, 5)$; $B \equiv (4, 7)$. Find the coordinates of the points of trisection of seg AB.

(4) $A \equiv (-2, -6)$; $B \equiv (10, 14)$. Find the coordinates of the points which divide seg AB into 4 equal parts.

(5) $A \equiv (-2, -1)$; $B \equiv (3, -2)$. Seg AB is extended to P such that $2 \cdot l(AP) = 3 \cdot l(BP)$. Find the coordinates of P.

(6) Seg AB is the diameter of a circle whose centre is C. If $A \equiv (-2, 4)$ and $C \equiv (-3, -5)$, find the coordinates of B.

(7) (3, −5); (−4, 7); (−2, −1) are the mid points of the sides of a Δ. Find the coordinates of the vertices of the Δ.

(8) The point P (7, a) lies on the line joining A (−5, 2) and B (3, 6). Find the ratio $\frac{l(AP)}{l(PB)}$ and hence find a.

(9) Find the ratio in which the x and y axes divide the join of (−3, −2) and (4, −4).

(10) In what ratio does the y axis divide the join of (3, 5) and (6, 7)? Is the division internal or external? Find the coordinates of the point of division.

(11) The join of A (4, 3) and B (−3, b) cuts the x axis at P (1, 0) and the y axis at Q. Find b and the ratio $\frac{l(AQ)}{l(QB)}$.

(12) Find the coordinates of the points which divide internally and externally the join of $(p+q, p-q)$ and $(p-q, p+q)$ in the ratio $p : q$.

(13) If ABCD is a parallelogram and A≡(−2, 4); B≡(−3, 5) and D≡(2, −2), find the coordinates of C.

(14) Show that ABCD is a square if A≡(4, −1); B≡(6, 0); C≡(7, −2); and D≡(5, −3).

(15) Show that PQRS is a rectangle if P≡(2, 1); Q≡(−1, 3) R≡(−5, 3) and S≡(−2, −5).

(16) Show that ABCD is a rhombus if A≡(2, 4); B≡(1, 9); C≡(6, 10) and D≡(7, 5).

(17) A, B, C are 3 points such that seg AB is bisected by x axis, seg AC is bisected by y axis and seg BC is bisected at the origin. If A≡(6, 9), find the coordinates of B and C.

CHAPTER 3

Locus

§ 3·1. The concept of locus :

In this chapter, we are concerned with finding a set of points which satisfy certain geometrical conditions, e. g.

Ex. (i) Suppose we are given a fixed point A and we require the set of all points which are at a distance of 5 units from A, then obviously all points lying on the circumference of the circle with center A and radius 5 units will satisfy the given condition. Hence the required set of points is the set of all points on the circle with centre A and radius 5.

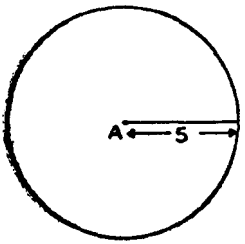


Fig. 3·1

Ex. (ii) Suppose A and B are two fixed points and we want the set of all points which are equidistant from A and B, then obviously every point on the perpendicular bisector of seg AB will satisfy the given condition and hence the required set of points is the set of all points on the perpendicular bisector of seg AB.

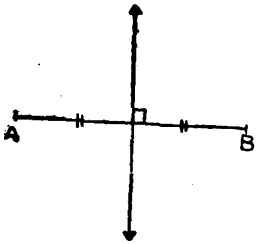


Fig. 3·2

Hence given a geometrical condition, we are able to find a set of points which satisfy this condition. Such a set of points is called a **locus.**

Definition : Locus : A *locus* is a set of points which satisfy a given geometrical condition.

Hence in Ex. (i) we can say that the locus of points which are at a distance of 5 units from A is the circle with centre A and radius 5. In Ex. (ii) we can say that the locus of points equidistant from A and B is the perpendicular bisector of seg AB.

To find a locus geometrically in this manner [as done in Ex. (i) and (ii)] is not always possible. If the given condition is such

that it cannot be interpreted geometrically, we cannot represent the locus geometrically.

e. g. If A and B are given points and we want the locus of a point such that the sum of its distances from A and B is 4 units then obviously it is not easilypossible to find geometrically the set ofpoints represented by this locus. There are many such conditions where a geometrical representation of the locus is not easily possible. Here coordinate geometry will help us in reducing a geometrical representation into an algebraic one. Let us therefore find out how a locus can be represented algebraically.

§ 3·11. Equation of a locus:

Let us consider the Ex. (i) given in § 3·1. Since A is a fixed point, it will have some fixed coordinates. Suppose $A \equiv (1, 2)$. Now let P be any point of the required locus. $\therefore$ $l(PA) = 5$. Suppose P has coordinates (x, y). Since $l(PA) = 5$, we have

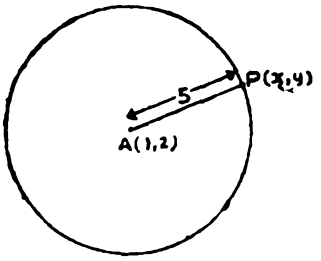


Fig. 3·3

$$\sqrt{(x-1)^2+(y-2)^2} = 5$$

$$\therefore (x-1)^2+(y-2)^2 = 25 \quad \ldots(i)$$

Now this equation must be satisfied by the coordinates of any point (x, y) belonging to this locus, e. g. if we want to find out whether the point (4, 6) belongs to this locus or not, we put $x=4$ and $y=6$ in the L. H. S. of (i) and obtain

L. H. S. $= (4-1)^2+(6-2)^2 = 9+16 = 25 =$ R. H. S.

Hence (4, 6) is a point of the locus. Similarly, we can conclude that (3, 4) is not a point of the locus since if we put $x=3$ and $y=4$ in the L. H. S. of (i), we get

L. H. S. $= (3-1)^2+(4-2)^2 = 4+4 = 8 \neq$ R H. S.

Hence the required locus is the set of all points whose coordinates satisfy equation (i) or in the set builder notation, we can say : The required locus $\equiv \{(x, y)/(x-1)^2+(y-2)^2 = 25\}$.

This gives us the algebraic representation of a locus. The locus will be represented by an equation involving x, y where (x, y) represents any point on this locus. If a point belongs to the locus, its coordinates must satisfy the equation of the locus and vice versa. Also if the coordinates of a point do not satisfy

the equation of the locus, the point does not belong to the locus. Hence we have :

Definition : Equation of a locus : If (x, y) is any point on a locus and if an equation involving x, y and no other unknowns is found from the given condition of the locus, this equation is called as the equation of the locus.

Hence the method of finding the equation of a locus consists of the following steps : (i) select (x, y) as any point of the locus, (ii) write the condition of the locus geometrically, (iii) convert this condition into an equation involving x, y and simplify this equation.

Let us see how this is done in examples :

§ 3.12. **Examples :**

Ex. 1. A (3, 2) and B ($-5, 6$) are two points. Find the equation of the locus of a point which is equidistant from A and B. Check whether the points (1, 8) and ($-2, 5$) belong to the locus.

Ans. Let P (x, y) be any point on the locus. Since P is equidistant from A (3, 2) and B ($-5, 6,$), we have

$$l(\text{PA}) = l(\text{PB})$$

$$\therefore \quad \sqrt{(x-3)^2+(y-2)^2} = \sqrt{(x+5)^2+(y-6)^2}$$

$$\therefore \quad x^2-6x+9+y^2-4y+4 = x^2+10x+25+y^2-12y+36$$

$$\therefore \quad -6x-4y+13 = 10x-12y+61$$

$$\therefore \quad 16x-8y+48 = 0$$

$$\therefore \quad 2x-y+6=0 \quad \ldots\ldots (\text{i})$$ is the equation of the locus.

To check whether (1, 8) belongs to the locus, putting $x=1$, $y=8$ in the L. H. S. of (i) we get

$$\text{L. H. S.} = 2(1)-8+6=0=\text{R. H. S.}$$

$\therefore$ (1, 8) belongs to the locus.

Similarly for the point ($-2, 5$), put $x=-2$ and $y=5$ in the L. H. S. of (i)

$\therefore$ L. H. S. $= 2(-2)-5+6=-3 \neq$ R. H. S.

$\therefore$ ($-2, 5$) does not belong to the locus.

Ex. 2. Find the equation of the locus of a point whose distance from the point (3, 2,) is twice its distance from the point (2, 1). Also find the points where the locus meets the x axis.

Ans. Let P (x, y) be any point on the locus.

Let A $\equiv (3, 2)$ and B $\equiv (2, 1)$

$\therefore$ From the given condition, $l(PA) = 2 \cdot l(PB)$.

$\therefore \sqrt{(x-3)^2+(y-2)^2} = 2 \cdot \sqrt{(x-2)^2+(y-1)^2}$

Squaring, we get:

$(x^2-6x+9+y^2-4y+4) = 4(x^2-4x+4+y^2-2y+1)$

$\therefore x^2+y^2-6x-4y+13 = 4x^2+4y^2-16x-8y+20$

$\therefore 3x^2+3y^2-10x-4y+7 = 0$ is the equation of the locus.

To find the points where the locus meets the x axis, we note that such a point will have its y coordinate zero. Hence we want to find out those points on the locus whose y coordinate is zero. Hence putting $y = 0$ in the equation of the locus we get

$$3x^2-10x+7=0$$

$$\therefore (3x-7)(x-1)=0 \qquad \therefore x=1 \text{ or } \frac{7}{3}$$

Hence the points where the locus meets the x-axis are $(1, 0)$ and $\left(\frac{7}{3}, 0\right)$.

Ex. 3. If A $(3,-2)$ and B $(-1, 4)$ are two points, find the locus of a point P such that AB subtends a right angle at P.

Ans. Let P (x, y) be any point on the locus. Since AB subtends a right angle at P, the triangle APB is right angled at P.

$\therefore [l(PA)]^2+[l(PB)]^2=[l(AB)]^2$

$\therefore (x-3)^2+(y+2)^2+(x+1)^2+(y-4)^2$
$=(3+1)^2+(-2-4)^2$

$\therefore x^2-6x+9+y^2+4y+4+x^2+2x+1+y^2-8y+16$
$=16+36$

$\therefore 2x^2+2y^2-4x-4y-22=0$

$\therefore x^2+y^2-2x-2y-11=0$ is the equation of the locus.

Ex. 4. If A $(6, 0)$ and B $(-6, 0)$ are two points, show the equation of the locus of a point, the sum of whose distances from A and B is 20 is given by $\frac{x^2}{100}+\frac{y^2}{64}=1$.

Ans. Let P (x, y) be any point on the locus. Since $(PA)+l(PB)=20$

$\therefore \sqrt{(x-6)^2+(y-0)^2}+\sqrt{(x+6)^2+(y-0)^2}=20$

$\therefore \sqrt{x^2+y^2-12x+36}+\sqrt{x^2+y^2+12x+36}=20$

$\therefore \sqrt{x^2+y^2-12x+36}=20-\sqrt{x^2+y^2+12x+36}$

(Note this step).

Squaring both sides, we get

$$x^2+y^2-12x+36=400-40\sqrt{x^2+y^2+12y+36}$$
$$+(x^2+y^2+12x+36)$$

$$\therefore\ -12x=400-40\sqrt{x^2+y^2+12x+36}+12x$$

$$\therefore\ 40\sqrt{x^2+y^2+12x+36}=400+24x \quad \text{(Note this step)}$$

$$\therefore\ 5\sqrt{x^2+y^2+12x+36}=50+3x$$

Squaring both sides, we get

$$25(x^2+y^2+12x+36)=2500+300x+9x^2$$

$$\therefore\ 25x^2+25y^2+300x+900=2500+300x+9x^2$$

$$\therefore\ 16x^2+25y^2=1600$$

$$\therefore\ \frac{16x^2}{1600}+\frac{25y^2}{1600}=1$$

$$\therefore\ \frac{x^2}{100}+\frac{y^2}{64}=1 \text{ is the required equation of the locus.}$$

Ex. 5. The equation of a locus is $y^2=ax$. If the points (8, 8) and (2, b) lie on the locus, find a and b.

Ans. Since the point (8, 8) lies on the locus $y^2=ax$, putting $x=8$ and $y=8$, we get $64=8a$ $\therefore\ a=8$.

$\therefore$ The equation of the locus becomes $y^2=8x$.

Since the point (2, b) belongs to the locus $y^2=8x$, putting $x=2$ and $y=b$, we get $b^2=16$ $\therefore\ b=\pm 4$.

Hence $a=8$ and $b=\pm 4$.

Ex. 6. Find the equation of the locus of a point whose distance from the x-axis exceeds its distance from y axis by 4.

Ans. Let P(x, y) be any point of the locus.

$\therefore$ Distance of P from x-axis $=y$ and
distance of P from y-axis $=x$.

We are given that y exceeds x by 4

$\therefore\ y-x=4$ is the equation of the locus.

Exercise 3·1 :

(1) Fill in the blanks :—

(a) The equation of a locus is $x+y=2$. $\therefore$ It meets the x axis at (———, ———).

(b) The equation of a locus is $x-2y=k$. If (3, −4) lies on it, $k=$———.

(c) The equation of a locus is $x^2+y^2=13$. If (k, 3) lies on it, $k=$——— or ———.

(d) P(x, y) is any point of a locus such that the distance of P from x axis is twice its distance from y axis.
$\therefore$ Equation of the locus is ———.

(e) P (x, y) is any point of a locus such that the distance of P from y axis exceeds its distance from x axis by 7.
$\therefore$ The equation of the locus is ——.

(2) Find the equation of the locus of a point whose distance from the x axis is half its distance from the origin.

(3) Find the equation of the locus of a point whose distance from y axis is three times its distance from the origin.

(4) Find the equation of the locus of a point whose distance from $(2, -3)$ is always 5 units. Also find the points where the locus meets the x axis.

(5) Find the equation of the locus of a point which is equidistant from the points $(3, -4)$ and $(-1, 8)$. Check whether the points $(1, 2)$ ond $(3, -5)$ lie on the locus.

(6) Find the equation of the locus of a point whose distance from $(-2, 1)$ is thrice its distance from $(1, 4)$.

(7) Find the equation of the locus of a point P such that the join of $(-2, 3)$ and $(6, -7)$ subtends a right angle at P.

(8) A $(1, -2)$; B $(-2, 4)$ and C $(3, 3)$ are given points. Find the equation of the locus of a point P such that $[l(PA)]^2 + [l(PB)]^2 = [l(PC)]^2$.

(9) The coordinates of A and B are $(3, -1)$ and $(2, 4)$ respectively. Find the equation of the locus of P if $2 \cdot l(PA) = 3 \cdot l(PB)$.

(10) Find the equation of the locus of a point such that the sum of the squares of its distances from the points $(3, 0)$ and $(0, -4)$ is 12.

§ 3.13. Examples:

Ex. 1. A and B are variable points on the x and y axes respectively such that $l(AB) = 4$. Find the locus of the point which divides seg AB internally in the ratio 1 : 2.

Ans. Let A and B lie on x and y axis respectively, such that $l(AB) = 4$. Let P divide AB internally in the ratio 1 : 2. We have to find the locus of P. Let $P \equiv (x, y)$, $A \equiv (a, 0)$ and $B \equiv (0, b)$. Since $l(AB) = 4$,

$\therefore \quad a^2 + b^2 = 16 \quad \ldots\ldots \quad (i)$

Now since P (x, y) dlvide s the join of A $(a, 0)$ and B $(0, b)$ internally in the ratio 1 : 2, we have

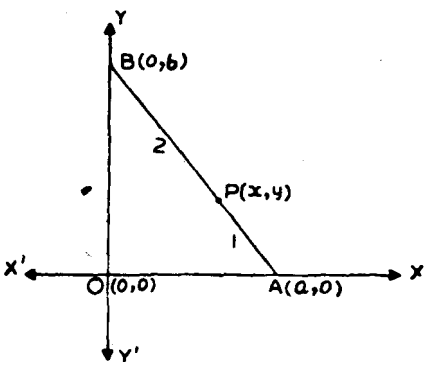


Fig. 3·4

$$x = \frac{1(0)+2(a)}{1+2} = \frac{2a}{3}$$

and $$y = \frac{1(b)+2(0)}{1+2} = \frac{b}{3}.$$

$$\therefore \quad a = \frac{3x}{2} \quad \text{and} \quad b = 3y.$$

Putting these values in (i), we get $\left(\frac{3x}{2}\right)^2 + (3y)^2 = 16$

$\therefore$ $9x^2 + 36y^2 = 64$ is the equation of the locus of P.

Ex. 2. A (1, 2) is a fixed point. A point B lies on a locus whose equation is $x^2 + y^2 = 4$. Find the locus of the midpoint of seg AB.

Ans. A (1, 2) is given point. Let B be one position of the variable point on the locus $x^2 + y^2 = 4$. Let P be the midpoint of seg AB. We have to find the locus of P. Let P $\equiv (x, y)$ and B $\equiv (h, k)$ Since B (h, k) lies on the locus $x^2 + y^2 = 4$;

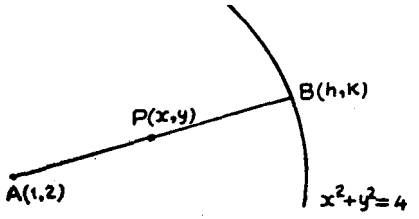


Fig. 3·5

$\therefore$ $h^2 + k^2 = 4$ .... (i)

Since P (x, y) is the mid point of A (1, 2) and B (h, k), we have

$$x = \frac{1+h}{2} \; ; \quad y = \frac{2+k}{2}.$$

$\therefore$ $h = 2x - 1$ and $k = 2y - 2$

Putting these values in (i),

$$(2x-1)^2 + (2y-2)^2 = 4$$

$\therefore$ $4x^2 - 4x + 1 + 4y^2 - 8y + 4 = 4$

$\therefore$ $4x^2 + 4y^2 - 4x - 8y + 1 = 0$ is the equation of the locus of P.

Ex. 3. A (3, 2) and B (-1, 4) are fixed points. C is a variable point on the locus whose equation is $x - 3y = 4$. Find the locus of the centroid of triangle ABC.

Ans. Let A (3, 2) and B (− 1, 4) be the given points Let C be one position of the variable point on the locus $x+3y=4$. Join AB, BC and CA. Let G be the centroid of triangle ABC. We have to find the locus of G.

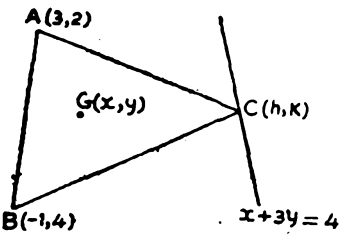


Fig. 3·6

Let $G \equiv (x, y)$ and $C \equiv (h, k)$. Since C (h, k) lies on the locus $x+3y=4$,

$\therefore\ h+3k=4 \ldots\ (i)$

Since G is the centroid of triangle ABC, we have,

$$x=\frac{3+(-1)+h}{3} \quad \text{and} \quad y=\frac{2+4+k}{3}.$$

$\therefore\ h=3x-2$ and $k=3y-6$.

Putting these values in (i)

$$(3x-2)+3\,(3y-6)=4$$
$$\therefore\quad 3x-2+9y-18=4$$
$$\therefore\quad 3x+9y-24=0$$

$\therefore\ x+3y-8=0$ is the equation of the locus of G.

Exercise 3·2:

(1) A and B are variable points on the x and y axis respectively. If l (AB) = 5, find the locus of the mid point of seg AB.

(2) A and B are variable points on the x and y axis respectively such that $2 \cdot l$ (OA) = $3 \cdot l$ (OB), O being the origin. Find the locus of the point which divides seg AB internally in the ratio 2 : 3.

(3) A and B are variables points on the x and y axis respectively such that $4 \cdot l$ (OA) = $3 \cdot l$ (OB), O being the origin. Find the locus of the point which divides seg AB internally in the ratio 2 : 1.

(4) A ≡ (−2, 0) and P is any point on the locus whose equation is $y^2 = 16x$. If Q bisects seg AP, find the locus of Q.

(5) A ≡ (3, −1) and B is any point on the locus $x^2+y^2=20$. If seg AB is extended to P such that l (AP) = $3 \cdot l$ (AB), find the locus of P.

(6) A ≡ (−3, 0). P is any point on the locus whose equation is $y^2=8x$. Q divides seg AP internally in the ratio 3 : 2. Find the equation of the locus of P.

(7) A ≡ (−4, −3) and Q is any point on the curve whose equation is $y=9x^2+47x+56$. P is taken on seg AQ produced such that l (AP) = $3 \cdot l$ (AQ). Show that the equation of the locus of P is $y=3x^2-x-10$.

(8) A (2, 5) and B (4, −11) are two fixed points. C is any point on the locus $3x+4y+5=0$. Find the locus of the centroid of triangle ABC.

(9) A is any point on the locus whose equation is $x^2+y^2-6x-4y=0$. B (4, 5) is a given point. Find the equation of the locus of the centroid of triangle OAB where O is the origin.

(10) The centroid of the triangle ABC always lies on the locus $y=1+x^2$. If $B\equiv(4, -3)$ and $C\equiv(0, 2)$, find the equation of the locus of A.

§ 3·2. Equation of a circle :

A circle is a very important locus. Since the distance from the centre of any point on a circle is constant, we define a circle as follows :

Definition : ' A *circle* is the locus of a point whose distance from a fixed point is always constant.

The fixed point is called the *centre* and the fixed distance is called the *radius* of the circle.

To find the equation of a circle, the coordinates of its centre and the length of its radius are required. We can then establish the equation of the circle as follows :

Let C (h, k) be the centre of a circle and r be its radius. Let P (x, y) be any point on the circle.

By the definition of a circle we have l (PC) $= r$

$\therefore \quad \sqrt{(x-h)^2+(y-k)^2} = r$

$\therefore \quad (x-h)^2+(y-k)^2=r^2$ is the equation of a circle with centre (h, k) and radius r, e. g. If the centre is (2, 3) and the radius is 4, the equation of the circle is $(x-2)^2+(y-3)^2=(4)^2$

i. e. $x^2-4x+4+y^2-6y+9=16$

$\therefore \quad x^2+y^2-4x-6y-3=0$ is the equation of the circle.

§ 3·21. Examples :

Ex. 1. Find the equation of a circle whose centre is $(3, -1)$ and which passes through the point $(-1, 2)$.

Ans. Let $C \equiv (3, -1)$ and $A \equiv (-1, 2)$

Since C is the centre of the circle and A is a point on the circle, l (CA) = radius of the circle.

$\therefore$ Radius $= l$ (CA) $= \sqrt{(3+1)^2-(-1-2)^2} = 5$

Hence the centre of the circle is C $(3, -1)$ and its radius is **5.**

$\therefore$ Its equation is $(x-3)^2+(y+1)^2=(5)^2$

$\therefore \quad x^2-6x+9+y^2+2y+1=25$

$\therefore \quad x^2+y^2-6x+2y-15=0$ is the equation of the circle.

Ex. 2. Find the equation of the circle drawn on seg AB as diameter if A $(-1, 4)$ and B $\equiv (3, -6)$.

Ans. Since seg AB is the diameter of the circle, the centre of the circle is the mid point of seg AB.

$$\therefore \quad \text{Centre} \equiv \left(\frac{-1+3}{2}, \frac{4-6}{2}\right) \equiv (1, -1)$$

$$\text{Also } l(AB) = \sqrt{(-1-3)^2+(4+6)^2}$$

$$= \sqrt{16+100} = \sqrt{116}$$

i. e. length of diameter $= \sqrt{116} = 2\sqrt{29}$

$\therefore$ length of radius $= \sqrt{29}$

Hence the circle has centre $(1, -1)$ and radius $\sqrt{29}$.

$\therefore$ Its equation is $(x-1)^2+(y+1)^2=(\sqrt{29})^2$

$\therefore \quad x^2-2x+1+y^2+2y+1=29$

$\therefore \quad x^2+y^2-2x+2y-27=0$ is the equation of the circle.

Exercise 3·3 :

(1) Fill in the blanks :

(a) The equation of a circle with centre $(2, 1)$ and radius 5 is ———.

(b) The equation of a circle with centre $(0, 0)$ ane radius 4 is ———.

(c) The equation of a circle with centre $(-2, 5)$ and radius $\sqrt{8}$ is ———.

(d) The equation of a circle with centre $(-3, -4)$ and radius $\sqrt{11}$ is ———.

(e) The equation of a circle is $(x-5)^2+(y+1)^2=6$.
$\therefore$ Its centre is (——, ——) and radius = ———.

(f) The equation of a circle is $x^2+(y-5)^2=25$.
$\therefore$ Its centre is (——, ——) and radius = ———.

(g) The equation of a circle is $(x+2)^2+y^2=36$
$\therefore$ Its centre is (——, ——) and radius = ———.

(2) Find the equation of a circle with centre $(3, -1)$ if it passes through $(8, 11)$.

(3) Find the equation of a circle with centre $(-1, -2)$ if it passes through the origin.

(4) Find the equation of a circle with centre at the origin and which passes through $(5, -1)$.

(5) Find the equation of a circle drawn on the line joining $(-1, 4)$ and $(3, 2)$ as diameter.

(6) Find the equation of a circle drawn on the line joining $(3, 4)$ and $(0, 0)$ as diameter.

(7) Find the equation of the circumcircle of the triangle whose vertices are $(0, 0)$; $(4, 0)$ and $(0, 6)$.

§ 3·3. Change of Origin :

Consider a circle whose centre is $(0, 0)$ and radius is r. The equation of this circle is $(x-0)^2 + (y-0)^2 = r^2$, i. e. $x^2+y^2=r^2$. This equation is very simple compared to the general equation $(x-h)^2+(y-k)^2=r^2$ of a circle with centre at (h, k) and radius r. The reason for this is that the centre of the circle $x^2+y^2=r^2$ is the origin. Hence if we choose the origin at the centre of a circle, the equation is simplified. This operation of changing the origin to a convenient point is very important in coordinate geometry. Let us find out how this is done.

§ 3·31. Formula for change of origin :

Consider a system of coordinate axes X'OX and Y'OY. Let $P(x, y)$ be any point in the plane. Suppose we wish to choose a point $O'(h, k)$ as the new origin and take the new axes $X_1'O'X_1$ and $Y_1'O'Y_1$ parallel to the old axes. Then with reference to this new system of coordinate axes, we must find the new coordinates of P.

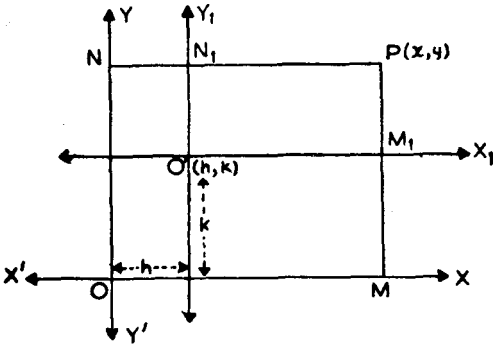


Fig. 3·7

Let the new coordinates of P be (x', y'). Draw PM perpendicular to OX, meeting OX at M and $O'X_1$ at M_1. Draw PN perpendicular to OY meeting OY at N and $O'Y_1$ at N_1. Then $l(PN)=x$; $l(PN_1)=x'$ and $l(NN_1)=h$.

Since $l(PN_1) = l(PN) - l(NN_1)$, $\therefore x' = x - h$.

Similarly, $l(PM) = y$, $l(PM_1) = y'$ and $l(MM_1) = k$ and since $l(PM_1) = l(PM) - l(MM_1)$ $\therefore y' = y - k$.

Hence if the coordinates of a point are (x, y) and they change to (x', y') when origin is changed to (h, k), then $x' = x - h$ and $y' = y - k$.

§ 3·31. Examples :

Ex. 1. If the origin is shifted to $(3, 4)$, the axes remaining parallel, find the new coordinates of the points $(4, 5)$, $(3, -1)$, $(-2, 4)$, $(-3, -4)$.

Ans. On shifting the origin to $(3, 4)$, let the new coordinates of (x, y) be (x', y'), $\therefore x' = x - 3$, $y' = y - 4$.

Hence new coordinates of f (x, y) are $(x - 3, y - 4)$.

$\therefore$ New coordinates of $(4, 5)$ are $(4 - 3, 5 - 4)$, i.e. $(1, 1)$.

New coordinates of $(3, -1)$ are $(3 - 3, 1 - 4)$, i.e. $(0, -5)$.

New coordidates of $(-2, 4)$ are $(-2 - 3, 4 - 4)$, i. e. $(-5, 0)$. and new coordinates of $(-3, -4)$ are $(-3 - 3, -4 - 4)$

i. e. $(-6, -8)$.

Ex. 2. When the origin is shifted to $(-1. 4)$, the axes remaining parallel, the new coordinates of three points are $(3, 2)$, $(4, -3)$, $(-1, 7)$. Find their old coordinates.

Ans. On shifting the origin to $(-1, 4)$, let the new coordinates of (x, y) be (x', y').

Then $x' = x - (-1) = x + 1$ $\therefore x = x' - 1$

and $y' = y - 4$ $\therefore y = y' + 4$.

Hence old coordinates of (x', y') are $(x' - 1, y' + 4)$;

$\therefore$ Old coordinates of $(3, 2)$ are $(3 - 1, 2 + 4.) \equiv (2, 6)$,

old coordinates of $(4, -3)$ are $(4, -1, -3 + 4) \equiv (3, 1)$ and old coordinates of $(-1, 7)$ are $(-1 \ -1, 7 + 4) \equiv (-2, 11)$.

Ex. 3. On shifting the origin to a point (h, k) axes remaining parallel, the coordinates of $(-2, 1)$ become $(4, -5)$. Find h and k.

Ans. On shifting the origin to (h, k), the coordinates (x, y) become $(x - h, y - k)$

$\therefore$ New coordinates of $(-2, 1)$ will be $(-2 - h, 1 - k)$.

But the new coordinates are $(4, -5)$,

$\therefore \quad -2-h=4$ and $1-k=-5 \quad \therefore \quad h=-6$ and $k=6$.

$\therefore$ The origin was shifted to $(-6, 6)$.

Ex. 4. The equation of a locus is $y^2+8y-4x+24=0$. If the origin is shifted to $(2, -4)$, the axes remaining parallel, find the new equation of the locus.

Ans. Suppose (x, y) changes to (x', y') when origin is changed to $(2, -4)$.

Then $x'=x-2$ and $y'=y+4$

$\therefore \quad x=x'+2$ and $y=y'-4$

Putting these values in $y^2+8y-4x+24=0$ we get

$$(y'-4)^2+8(y'-4)-4(x'+2)+24=0.$$

$\therefore \quad (y')^2-8y'+16+8y'-32-4x'-8+24=0$

$\therefore \quad (y')^2-4x'=0.$

Dropping the dashes, the new equation of the locus is $y^2=4x$.

Ex. 5. By shifting the origin to a suitable point, axes remaining parallel, if the equation $x^2+y^2-4x+6y-3=0$ becomes of the form $x^2+y^2=k^2$, find k.

Ans. Consider the equation

$$x^2+y^2-4x+6y-3=0$$

$\therefore \quad x^2-4x+y^2+6y=3$

$\therefore \quad x^2-4x+4+y^2+6y+9=3+4+9$

$\therefore \quad (x-2)^2+(y+3)^2=16$

$\therefore \quad (x-2)^2+[y-(-3)]^2=16 \qquad \ldots\ (i)$

Hence if the origin is shifted to $(2, -3)$ then; if (x, y) changes to (x', y'), we have $x'=x-2$, $y'=y-(-3)=y+3$. Hence shifting the origin to $(2, -3)$, the equation (i) becomes $x'^2+y'^2=16$ which is of the form $x^2+y^2=k^2$ where $k^2=16$

$\therefore \quad k=\pm 4.$

Exercise 3.4 :

(1) Fill in the blanks:

If origin is shifted to (2, −3), axes remaining parallel, then

(a) new coordinates of (x, y) are (——, ——);

(b) new coordinates of (3, −4) are (——, ——);

(c) new coordinates of (−2, 1) are (——, ——);

(d) new coordinates of (2, 4) are (——, ——);

(e) new coordinates of (4, −3) are (——, ——);

(f) new coordinates of (2, −3) are (——, ——).

(2) Fill in the blanks :

If origin is shifted to $(-4, 1)$, axes remaining parallel then

(a) old coordinates of (x', y') are (——, ——);

(b) old coordinates of $(2, -1)$ are (——, ——);

(c) old coordinates of $(-4, -3)$ are (——, ——);

(d) old coordinates of $(4, -1)$ are (——, ——);

(e) old coordinates of $(0, 0)$ are (——, ——);

(3) If origin is shifted to $(-2, 4)$, axes remaining parallel, $(4, b)$ lies on the new x axis and $(a, -5)$ lies on the new y-axis. Find a and b.

(4) If the new coordinates of $(3, 4)$ are $(-1, 7)$ when origin is shifted to P, axes remaining parallel, find the old coordinates of P.

(5) If the origin is shifted to $(-3, 1)$, axes remaining parallel, find the new equation of the locus whose old equation is

$$x^2+3y^2+6x+10=0.$$

(6) The equation of a locus is $x^2-4x-6y-20=0$ If the origin is shifted to $(2, -4)$ axes remaining parallel, find the new equation of the locus.

(7) Shifting the origin to a suitable point, show that each of the following equations of loci reduce to the form $x^2+y^2=k^2$ and hence find k.

(a) $x^2+y^2-4x-6y+9=0$.

(b) $x^2+y^2+6x-8y=0$.

(c) $x^2+y^2+2x-8=0$.

(d) $x^2+y^2+6y-16=0$.

(e) $x^2+y^2+8x-2y-8=0$.

CHAPTER 4

Circular Functions

In this chapter, we shall discuss the representation of a real number on a unit circle. We already know that there is one-one correspondence between the set of real numbers and the set of points on a straight line and every real number can be represented by a point on a straight line. Every real number can also be represented by a point on the circumference of a given circle. Such representation of the real number system is known as *circular representation.* Let us see how this can be done :

§ 4·1. The Unit Circle :

We have learnt that the equation of the circle of unit radius with centre at the origin is $x^2+y^2=1$. This circle is a *Unit Circle.*

Let this Unit Circle intersect the positive x-axis at the point A $\therefore$ A$\equiv(1,0)$. This point A is called the *base point* or the *initial point.* It represents zero. Now if we take a point P along the circumference of the unit circle starting from A in the anti-clockwise direction, such that the length of the arc AP is 1, then the point P represents the real number 1. Similarly, the point P′ obtained by starting from the base point A but in the clockwise direction such that length of the arc AP′ = 1, then P′ represents the real number -1. We assume here that every arc of the circle has a length, say s, and this length s is a real number.

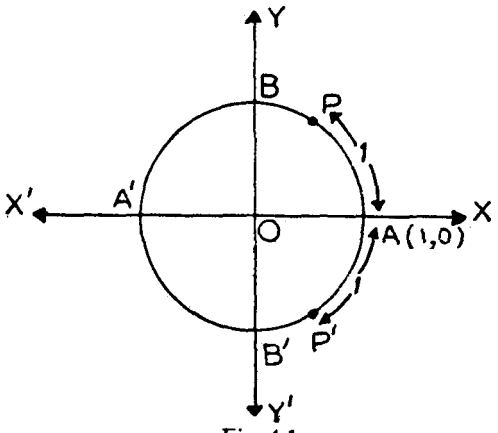


Fig. 4·1

Sign Convention : If the arc AP is measured along the unit circle in the anti-clockwise direction, the length of the arc is considered to be positive and if in clockwise direction, the length of the arc is considered to be negative.

If the unit circle intersects the x axis at the points A and A′ and y axis at the points B and B′ as shown in fig. 4·1, then similar to A $(1, 0)$ we have $A' \equiv (-1, 0)$, $B \equiv (0, 1)$, $B' \equiv (0, -1)$.

Since the circumference of this circle with radius $r = 1$ is $2\pi r = 2\pi \cdot 1 = 2\pi$, $\therefore$ the points A, B, A′ and B′ divide the circumference in four equal parts each of length $\frac{2\pi}{4} = \frac{\pi}{2}$. Hence, the lengths of arcs AB, AA′, AB′ are $\frac{\pi}{2}$, $\frac{2\pi}{2}$ and $\frac{3\pi}{2}$, i. e. $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ respectively. Thus the points B, A′ and B′ represent the real numbers $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ respectively.

Now if we start from the base point A, in the anticlokwise direction and measure the length $2\pi + 1$, (see fig. 4·2), then after one complete rotation (length 2π) we get the point P itself. $\therefore$ P also represents the real number $2\pi + 1$. Similarly P can also represent the real number $2(2\pi) + 1$, $3(2\pi) + 1$, ... and also $-2\pi + 1$, $2(-2\pi) + 1$, $3(-2\pi) + 1$, ... in general $n \cdot 2\pi + 1$ for all $n \in$ I. Similarly the point P′ represents the real number -1, $2\pi - 1$, $4\pi - 1$, $6\pi - 1$, ... , in general $2n\pi - 1$, for all $n \in$ I.

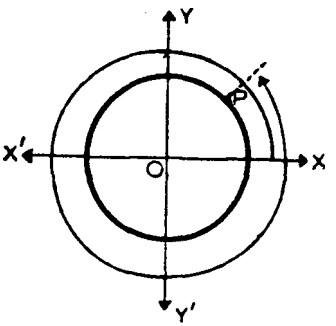


Fig. 4·2

Thus, corresponding to many real numbers such as -1, $\pm 2\pi - 1$, $\pm 4\pi - 1$, .. etc., we have only one point P′. Thus, in general, corresponding to many real numbers, we have only one point. Hence there exists *many-one correspondence* between the set of real numbers and the points on the unit circle. We recall here that between the set of real numbers and the points on a straight line, there is *one-one* correspondence, i. e. a point on a

straight line represents *one and only one* real number whereas, a point on the unit circle represents many real numbers.

e. g. if R is a point on the unit circle such that the length of the arc AR is s, then R represents a set of real numbers given by $2n\pi + s$, $n \in I$.

§ 4·11. Example :

Ex. 1. If A is (1, 0) and the points A, A_1, A_2, A_3, A_4, A_5, A_6 and A_7 divide the circumference of the unit circle in 8 equal parts, find the real numbers s where $s < 2\pi$ represented by these points. Further find, in general, the real numbers represented by the points A_2, A_4, A_6, and A.

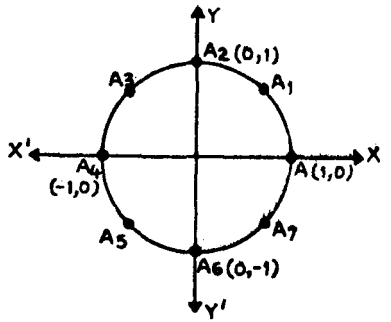


Fig. 4·3

Since these points divide the circumference 2π in 8 equal parts, each part is of length $\frac{2\pi}{8} = \frac{\pi}{4}$. $\therefore$ If we consider the real numbers s when $s < 2\pi$ the points A, A_1, A_2, A_3, A_4, A_5, A_6, A_7 represents the real numbers 0, $\frac{\pi}{4}$, $\frac{2\pi}{4}$, $\frac{3\pi}{4}$, $\frac{4\pi}{4}$, $\frac{5\pi}{4}$, $\frac{6\pi}{4}$, $\frac{7\pi}{4}$ i. e. 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{4}$.

In general, the point A_2 represents the real number $2\pi(n) + \frac{\pi}{2}$, i. e. $2n\pi + \frac{\pi}{2}$ for all $n \in I$. Similarly, the points A_4, A_6 and A represent, in general, the real numbers $(2n\pi + \pi)$, $\left(2n\pi + \frac{3\pi}{2}\right)$, $(2n\pi + 0)$, i. e. $(2n+1)\pi$, $\left(2n + \frac{3}{2}\right)\pi$, $2n\pi$ respectively.

Exercise 4·1 :

(1) If A is (1, 0) and the points A, A_1, A_2, A_3, A_4 and A_5 divide the circumference of the unit circle in *six* equal parts, find the real numbers s which correspond to these points.

(2) If A is (1, 0) and the points A, A_1, A_2, ...A_{11} divide the circumference of the unit circle in 12 equal parts, find the real numbers s which correspond to these points.

(3) A is (1, 0) and the points B, A′ and B′ represent the real numbers $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ respectively. Fill in the blanks in the following table :—

Real Number s	Corresponding point	Real Number s	Corresponding point
(I) $6\pi+\frac{\pi}{2}$	B	(II) $-5\frac{\pi}{2}$	B′
$3\pi+\frac{\pi}{2}$	—	$+7\frac{\pi}{2}$	—
$2\pi+\frac{3\pi}{2}$	—	$-15\frac{\pi}{2}$	—
$-2\pi+\frac{\pi}{2}$	—	-18π	—
-3π	—	$\frac{23\pi}{2}$	—
$+5\pi$	—	$\frac{-23\pi}{2}$	—
$2n\pi$	—	$n\pi$ (n even)	—
$(2n+1)\pi$	—	$n\pi$ (n odd)	—

§ 4·2. Circular Function :

We have already observed that there is many-one correspondence between the set of real numbers and the set of length (s) of arcs on the unit circle. Further with every point P on the unit circle, we can associate an ordered pair (x, y) whose components are the x and y coordinates of that point.

Thus, corresponding to every real number s, we have one and only one ordered pair (x, y) whose components are the x and y coordinates of the point P on the unit circle such that the length of the arc AP is s where A is (1, 0). Since corresponding to *many* real numbers $s + 2n\pi$ ($n \in$ I) we have the unique point P (x, y), i. e. corresponding to many real numbers $s+2n\pi$ ($n \in$ I), we have the unique ordered pair (x, y) which represents the x and y coordinates of P. Hence, there is *many-one* correspondence between the set of real numbers and the set of ordered pairs (x, y).

This many-one correspondence is a function and is known as a *Circular Function. Domain* of this circular function is the set of real numbers and the *Range* is the set of ordered pairs (x, y) whose components are the x and y coordinates of the points on the Unit Circle.

Definition : The circular function C is given by

$$C = \left\{ s, (x, y) \;\middle|\; \begin{array}{l}(x, y) \text{ are the coordinates of the} \\ \text{point P on the unit circle where A is} \\ (1, 0) \text{ and length of arc AP is } s.\end{array} \right\}$$

$$\therefore \quad C(s) = (x, y).$$

§ 4·21. **Examples :**

Ex. 1. With the usual notations find $C(0)$, $C\left(\frac{\pi}{2}\right)$, $C(\pi)$ and $C\left(\frac{3\pi}{2}\right)$.

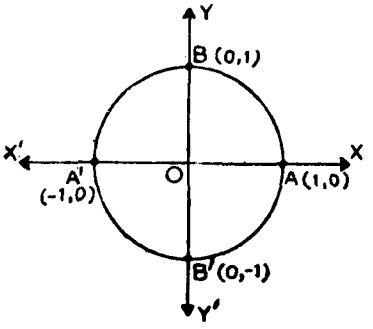


Fig. 4·4

Ans. The real numbers $0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ are represented by the points A, B, A′, B′ respectively as shown in fig. 4·4 and the co-ordinates of these points are $A(1, 0)$, $B(0, 1)$, $A'(-1, 0)$ and $B'(0, -1)$.

$$\therefore \quad C(0) = (1, 0), \; C\left(\frac{\pi}{2}\right) = (0, 1).$$

$$C(\pi) = (-1, 0), \; C\left(\frac{3\pi}{2}\right) = (0, -1).$$

§ 4.3. **Sine and Cosine functions:**

With the help of circular function C, we have $C(s) = (x, y)$. We define : x is called the *cosine* of s and y is called the *sine* of s and are denoted as $x = \cos s$ and $y = \sin s$, These functions are known as circular functions.

Let us find these for $s = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. From Ex. 1 in § 4·21 we have

$C(0) = (1, 0) \quad \therefore \quad \cos 0 = 1 \quad$ and $\quad \sin 0 = 0$

$C\left(\frac{\pi}{2}\right) = (0, 1) \quad \therefore \quad \cos \frac{\pi}{2} = 0 \quad$ and $\quad \sin \frac{\pi}{2} = 1$

$C(\pi) = (-1, 0)$ $\therefore \cos \pi = -1$ and $\sin \pi = 0$

$C\left(\frac{3\pi}{2}\right) = (0, -1)$ $\therefore \cos \frac{3\pi}{2} = 0$ and $\sin \frac{3\pi}{2} = -1$.

Further if we take a point P on the unit circle corresponding to the real number s then P is $(\cos s, \sin s)$.

With the help of the above relation we can define sine and cosine functions as follows :—

If $P(x, y)$ is a point on the unit circle corresponding to the real number s, then,

$$\text{cosine function} = \{(s, \cos s) \mid x = \cos s\}$$

$$\text{and sine function} = \{(s, \sin s) \mid y = \sin s\}.$$

In other words, if P is a point on the unit circle such that $AP = s$ where A is $(1, 0)$ then cosine function is the set of all ordered pairs $(s, \cos s)$ where $\cos s$ is the x coordinate of P and the sine function is the set of all ordered pairs $(s, \sin s)$ where $\sin s$ is the y coordinate of P.

Further since $x^2 + y^2 = 1$, we get $(\cos s)^2 + (\sin s)^2 = 1$, i. e. $\cos^2 s + \sin^2 s = 1$. [Note that square of $\cos s$ is written as $\cos^2 s$ and not $\cos s^2$; $\cos s^2$ denote the value of the cosine function corresponding to the real number s^2.]

Thus, for any real number θ, we can find $\cos \theta$ and $\sin \theta$ by finding the coordinates of the corresponding point on the unit circle and further if any one of them is given we can find the other by using $\cos^2 \theta + \sin^2 \theta = 1$, e. g. if $\sin \theta = \frac{\sqrt{3}}{2}$, $\cos^2 \theta = 1 - \sin^2 \theta$

$$= 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$\therefore \cos \theta = \pm \frac{1}{2}$.

Theorem 4·1 :

If $P(x, y)$ is the point on the unit circle corresponding to the real number θ and if A is $(1, 0)$, show that

$$[l(AP)]^2 = 2(1 - \cos \theta).$$

Proof : Since $P(x, y)$ reprssents the real number θ, we have $x = \cos \theta$, $y = \sin \theta$.

$\therefore$ $P \equiv (\cos \theta, \sin \theta)$, Also $A \equiv (1, 0)$

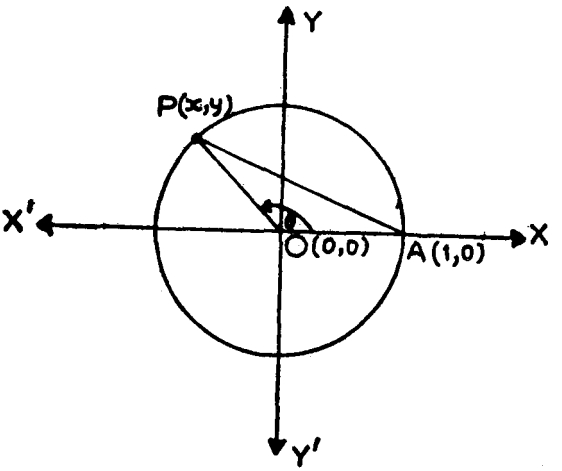


Fig. 4·5

$$\therefore \quad [l(AP)]^2 = (1-\cos\theta)^2 + (0-\sin\theta)^2$$
$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 1 - 2\cos\theta + 1$$
$$= 2 - 2\cos\theta = 2(1-\cos\theta)$$

Exercise 4.2 :

P (x, y) represents a real number s on the unit circle, i. e. $C(s) = (x, y)$ and $x = \cos s$ and $y = \sin s$ and $n \in N$. Using these, fill in the blanks :—

Real Number	P	(x, y)	$\cos s = x$	$\sin s = y$
0	A	(1, 0)	$\cos 0 = 1$	$\sin 0 = 0$
$\frac{\pi}{2}$	B	(0, 1)	$\cos\frac{\pi}{2} = 0$	$\sin\frac{\pi}{2} = 1$
π	A′	(−1, 0)	$\cos\pi =$ ——	$\sin\pi =$ ——
$\frac{3\pi}{2}$	B′	(0, −1)	$\cos\frac{3\pi}{2} =$ ——	$\sin\frac{3\pi}{2} =$ ——
2π	——	(,)	$\cos 2\pi =$ ——	$\sin 2\pi =$ ——
$2\pi + \frac{\pi}{2}$	——	(,)	$\cos\left(2\pi+\frac{\pi}{2}\right) =$ ——	$\sin\left(2\pi+\frac{\pi}{2}\right) =$ —
4π	A	(,)	$\cos 4\pi =$ ——	$\sin 4\pi =$ ——
6π	——	(,)	$\cos 6\pi =$ ——	$\sin 6\pi =$ ——
$2n\pi$	——	(,)	$\cos 2n\pi =$ ——	$\sin 2n\pi =$ ——
$n\pi$ (n even)	A	(,)	$\cos n\pi =$ —— (n even)	$\sin n\pi =$ —— (n even)
3π	——	(,)	$\cos 3\pi =$ ——	$\sin 3\pi =$ ——
5π	——	(,)	$\cos 5\pi =$ ——	$\sin 5\pi =$ ——
$(2n+1)\pi$	——	(,)	$\cos(2n+1)\pi =$ ——	$\sin(2n+1)\pi =$ ——
$n\pi$ (n odd)	A′	(,)	$\cos n\pi =$ —— (n odd)	$\sin n\pi =$ —— (n odd)

§ 4·31. Signs of Sine and Cosine functions :

cos s is positive or negative if x is positive or negative and similarly sin s is positive or negative if y is positive or negative. Thus, depending on the quadrant in which the point P lies, we have the positive or negative signs of sin s and cos s.

Quadrant II	*Quadrant I*
x — ve $\therefore \cos s$ — ve	x + ve $\therefore \cos s$ + ve
y + ve $\therefore \sin s$ + ve	y + ve $\therefore \sin s$ + ve
Quadrant III	*Quadrant IV*
x — ve $\therefore \cos s$ — ve	x + ve $\therefore \cos s$ + ve
y — ve $\therefore \sin s$ — ve	y — ve $\therefore \sin s$ — ve

§ 4·32. Domain and Range of Sine and Cosine functions :

Domain of sine and cosine functions is the set of real numbers.

Range : Since (x, y) are the coordinates of any point on the unit circle we have $x^2+y^2=1$. $\therefore$ neither x nor y can be greater than one. At the most, one of them can be one but in that case the other must be zero.

Since $x^2+y^2=1 \quad x^2 \leqslant 1$ and $y^2 \leqslant 1$

$x^2 \leqslant 1 \therefore |x| \leqslant 1 \therefore |\cos s| \leqslant 1 \therefore -1 \leqslant \cos s \leqslant 1$

$y \leqslant 1 \therefore |y| \leqslant 1 \therefore |\sin s| \leqslant 1 \therefore -1 \leqslant \sin s \leqslant 1.$

CHAPTER 5

Trigonometric Functions

§ 5·1. Angles :

In Geometry, an angle is defined to be the union of two rays with a common end point and that an angle can be designated by naming a point on each ray together with the common end point of the rays, or otherwise by assigning a symbol such as θ, α, β etc. to the angle. The common end point of the rays is called the vertex of the angle and the rays are called the sides of the angle.

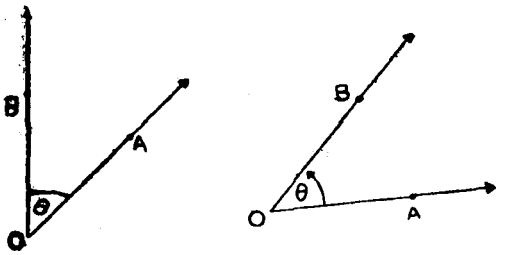


Fig. 5·1

For the angle θ shown in fig. 5·1, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the rays and O the common end point is the vertex and the angle is denoted by $\angle BOA$.

Thus in *Geometry* an angle is defined to be union of two rays with a common end point. In *Trigonometry*, we generalise this definition by introducing the measure of an angle which is the "amount of rotation" required to move from the position of the initial ray of an angle to the terminal ray. Thus for finding the measure of an angle, we require an initial side such as $\overrightarrow{OA}$ in fig. 5·1, a terminal side $\overrightarrow{OB}$ and the 'amount of rotation' and the angle is defined by the *ordered pair of rays*, such as $(\overrightarrow{OA}, \overrightarrow{OB})$ for the angle θ in fig. 5·1, *together with rotation* from $\overrightarrow{OA}$ to $\overrightarrow{OB}$. Hence

the ordered pair of rays $(\overrightarrow{OA},\overrightarrow{OB})$ and $(\overrightarrow{OB},\overrightarrow{OA})$ represent different angles because for the angle given by $(\overrightarrow{OA},\overrightarrow{OB})$, $\overrightarrow{OA}$ is the initial side, $\overrightarrow{OB}$ is the terminal side and the rotation is from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ whereas for the angle given by $(\overrightarrow{OB},\overrightarrow{OA})$ the rotation is from $\overrightarrow{OB}$ to $\overrightarrow{OA}$, $\overrightarrow{OB}$ being the initial side and $\overrightarrow{OA}$ being the terminal side. The common end point O of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is called the vertex of the angle. Thus in Trignometry, we define angles with the help of *ordered pairs of rays together with the rotation.*

There are two possible directions of rotation, viz. anticlockwise and clockwise, by which the position of the ray OA can be moved to the position of the ray OB. These directions of the rotation can be shown by drawing a curved arrow as shown in fig. 5·2 and fig. 5·3.

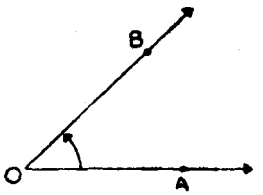


Fig. 5·2

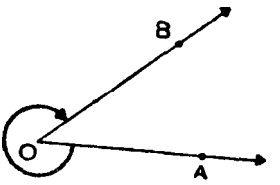


Fig. 5·3

If the rotation is anticlockwise as shown in fig. 5·2, the angle is said to be positive and if the rotation is clockwise as shown in fig. 5·3, the angle is said to be negative.

§ 5·11. Angle in standard position :

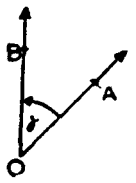


Fig. 5·4

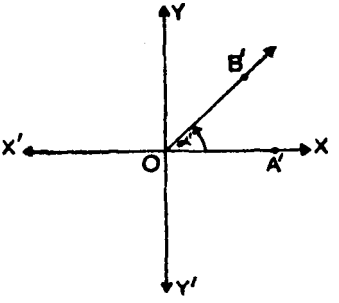


Fig. 5·5

When the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ of the angle AOB are in the cartesian plane, we can always find an angle A'OB' which is congruent to the angle AOB, i. e. which has the same measure as

the angle AOB in such a way that the vertex O is at the origin and the initial side OA is along the positive x axis. Such an angle is said to be in *standard position*. Thus, angle A'OB' in fig. 5·5 is in standard position and is congruent to (has the same measure as) angle AOB in fig. 5·4.

If the terminal side of an angle in standard position, is in a particular quadrant, the angle is said to be in that quadrant, e. g. in fig. 5·6, angle AOB is in first quadrant, angle COD is in second quadrant, angle EOF is in third quadrant and angle GOH is in fourth quadrant.

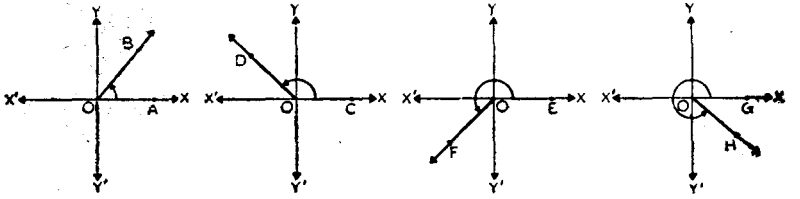


Fig. 5·6

The angle MON in fig. 5·7 is in the second quadrant just like the angle COD in fig. 5·6. But the angle COD in fig. 5·6 is positive because the rotation of the initial side as shown by the curved arrow is anticlockwise, whereas angle MON in fig. 5·7 is negative since the rotation is clockwise in this case.

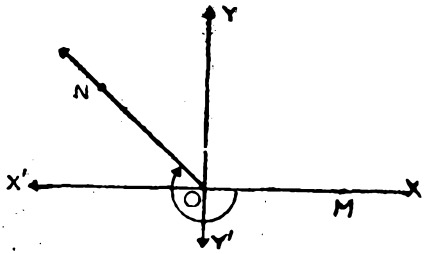


Fig. 5·7

Exercise 5·1 :

(1) State, with reasons, whether the angles given in fig. 5·8 are in standard position .

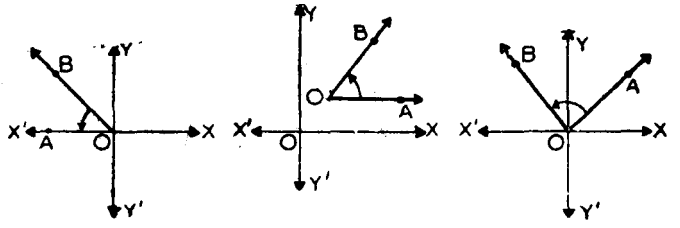


Fig- 5·8

(2) State the quadrants in which the angles in fig. 5·9 lie and state also whether they are positive or negative.

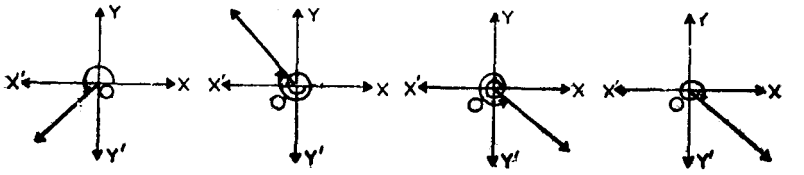


Fig. 5·9

§ 5·12. Coterminal Angles :

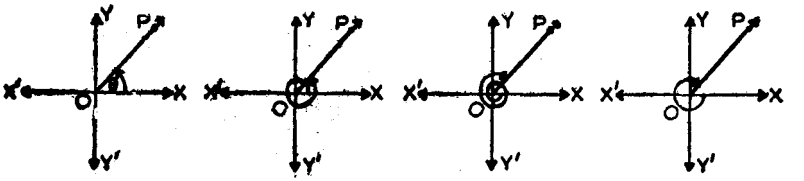


Fig. 5·10

All the angles θ. $360° + \theta$, $-720° + \theta$, $-360° + \theta$ shown in fig. 5·10 have the same initial and terminal side but their measures are different since the amount of rotation of each angle is different. Such angles are called *coterminal angles.*

In general, all the angles $n(360°) + \theta$ where $n \in I$ are coterminal angles.

§ 5·2. Angle Measurement :

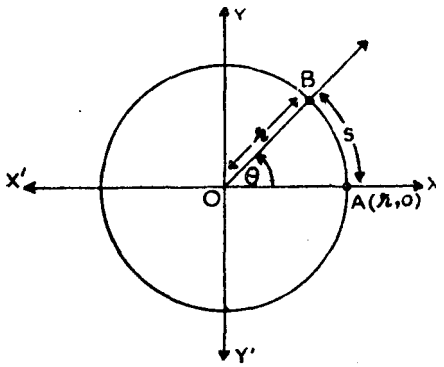


Fig. 5·11

A measure is assigned to an angle with the help of a circle with centre at the vertex O and of radius r, as follows :

§ 5·21. Measure in Rotation :

The most reasonable unit for measurement of an angle is the number of complete rotations or portions of complete rotation or both, which are required to move the initial side to the terminal side.

In Fig. 5·11, the angle AOB is in standard position with OA as the initial side and the terminal side intersecting the circle with centre O and radius r ($r > 0$) at the point B. If length of the arc AB is s then the measure of the angle AOB is defined as $\frac{s}{2\pi r}$. Thus, if the circumference is divided into 8 equal parts and C is the point in the first quadrant on the circle such that arc AC is one of these 8 equal parts as shown in Fig. 5·12, then the length of arc AC $= \frac{1}{8}$ (circumference) $= \frac{2\pi r}{8} = \frac{\pi r}{4}$. Hence the measure in *rotation* of the angle AOC is $\frac{1}{8}$ as given by $\frac{s}{2\pi r} = \frac{(\pi r)/4}{2\pi r} = \frac{1}{8}$. Similarly the measures in *rotation* of the angles AOB, AOA', AOB' are $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ respectively.

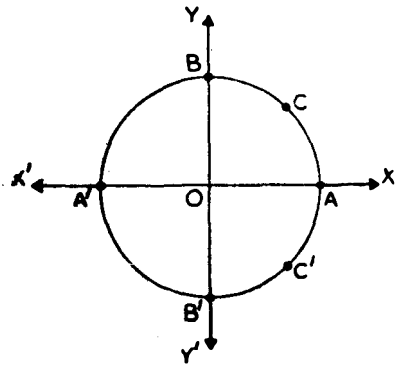


Fig. 5·12

It is customary to assign positive numbers to the measures of the angles where the rotation is in anticlockwise direction and negative numbers to the measures of the angles where the rotation is in the clockwise direction. Hence if lengths of arc AC and arc AC' are same, then measure in *rotation* of angle AOC' in the clockwise direction is $-\frac{1}{8}$. Similarly, the measures in *rotation* of angles AOB', AOA', AOB in clockwise direction are $-\frac{1}{4}$, $-\frac{1}{2}$ and $-\frac{3}{4}$ respectively.

Theorem 5·1. "The measure in rotation of any angle is independent of the radius of the measuring circle."

Proof: Let us consider two concentric circles as shown in fig, 5·13 with radii r and r', intersecting the initial side of the angle θ at A and A′ and the terminal side of this angle at B and B′ respectively. Thus, we have arc AB of length, say s, on the circle with radius r and arc A′B′ of length, say s' on the circle with radius r.

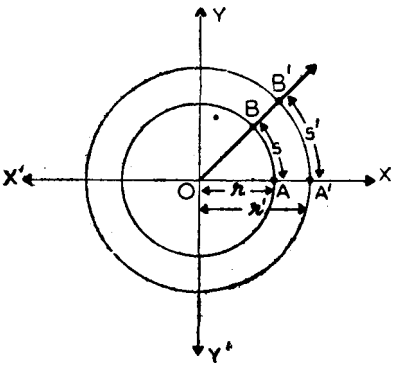


Fig. 5·13

But we know from Geometry that " the lengths of arcs intercepted by a given central angle on concentric circles are proportional to the circumference of the circles ".

Hence $\frac{s}{2\pi r} = \frac{s'}{2\pi r'}$. It follows therefore that " the measure in rotation of any angle is independent of the radius of the measuring circle ".

§ 5·22. Sexagesimal System :

One of the units of the measurement of an angle is a degree and the system in which an angle is measured in degrees is called ' sexagesimal system '.

A degree is defined as $\frac{1}{360}$ th part of one complete rotation. Hence the measure of an angle in degrees is the measure in rotations multiplied by 360°.

Thus, in a circle of radius r, if an arc of length s subtends an angle θ at the centre, then the measure of the angle in degrees $=$ measure in rotation $\times 360° = \frac{s}{2\pi r} \times 360°$.

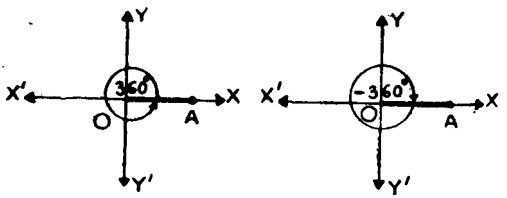


Fig. 5·14

Thus, the measure of an angle of one complete ratation in the anticlockwise direction is 360° and that in the clockwise direction is $-360°$. (see fig. 5·14)

Similarly, the measure of an angle of ' one-half ' of complete rotation ' is $+180°$ or $-180°$ depending on whether the direction is anticlockwise or clockwise. Such an angle is called a *straight angle.*

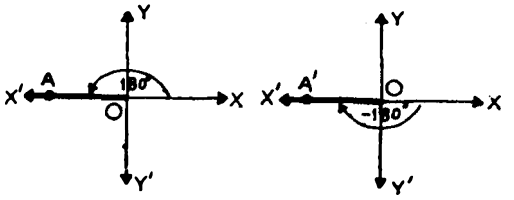


Fig. 5·15

The measure of an angle of ' one-fourth of complete rotation ' is $+90°$ or $-90°$ depending on whether the rotation is anticlockwise or clockwise. Such angles are called *right angles.*

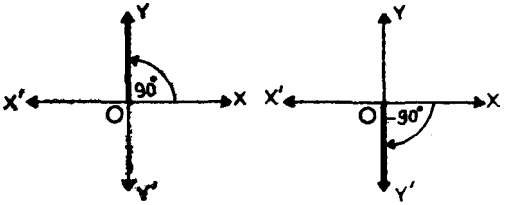


Fig. 5·16

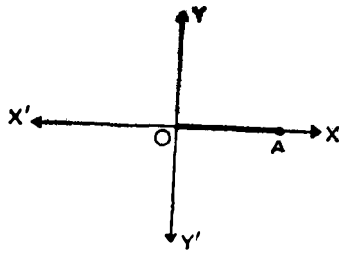


Fig. 5·17

If the terminal side of an angle coincides with initial side without any rotation, then the measure in rotation of such an angle is zero and its measure in degrees is $0 \times 360° = 0°$. This angle is called a *zero angle.*

With the help of the definition of an angle that we have in Trigonometry, the angles of two or three or more (say n, $n \in I$) complete rotations can also be defined and their measures in degrees are $\pm 2 \times 360°$, $\pm 3 \times 360°$,.... $\pm n \cdot 360°$.... the sign depending on the direction of rotation.

Thus, angles $A°$, $\pm 360° + A°$, $\pm 2 \cdot 360° + A°$.....
$\pm n \cdot 360° + A°$, $(n \in I)$.... have the same terminal side as that of angle $A°$. Such angle are said to be *equivalent angles.*

Further, angles with measures in degrees less than $1°$ can also be measured by subdividing a degree in minutes and seconds as follows :

One degree is $\frac{1}{360}$th part of one complete rotation. One minute is $\frac{1}{60}$th part of one degree and one second is $\frac{1}{60}$th part of one minute. An angle whose measure is x degrees, y minutes and z seconds is denoted as $x° \, y' z''$. Thus, an angle $45° \, 13' \, 27''$ denotes an angle whose measure is 45 degrees, 13 minutes and 27 seconds.

$\therefore \quad 1° = 60'$ and $1' = 60''$.

§ 5·23. Circular Measure :

The unit of measurement of an angle in this system is called a radian.

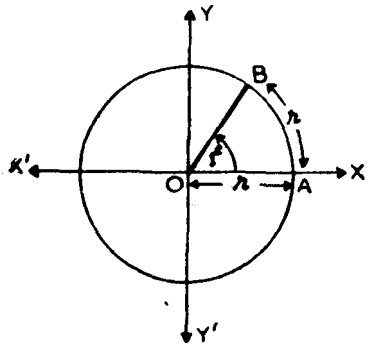


Fig. 5·18

Let us draw a circle of radius r ($r > 0$) with centre at the origin 0. Let it intersect the positive x axis at A. Let B be the point on the circumference of this circle, obtained by moving the point A in the anticlockwise direction such that the length of the arc AB is r. The circular measure of the angle AOB with initial side OA and terminal side OB is defined to be one *radian.* Hence we can say that a *radian* is an angle subtended at the center of a circle by an arc of the circle whose length is equal to the radius of the circle.

One radian is denoted as 1^c. *It is customary to omit this* ^{c} Thus an angle of measure 2 is understood to be an angle whose measure is 2 radians. Similarly $\sin \frac{\pi}{4}$ means sine of $\frac{\pi}{4}$ radians. Note that in case the measure of the angle is in deg rees, we must write the notation for degree, e. g. $\sin x°$ means sine of x degrees, whereas $\sin x$ means sine of x radians.

§ 5·24. Length of an arc in a circle :

We have seen that in a circle of radius r, an arc of length r subtends an angle 1^c at the centre of the circle. Now if an arc of length s subtends an angle θ^c at the centre, then we must find a relation between s, r and θ. This is done in the following theorem.

Theorem 5·2. If, in a circle of radius r, an arc of length s subtends an angle θ (measured in radians) at the centre, then $s = r\theta$.

Proof : Draw a circle with centre O and radius $r\ (r > 0)$. Select the points A, B, C, D on the circumference of this circle such that l (arc AB) = l (arc BC) = l (arc CD) = r.

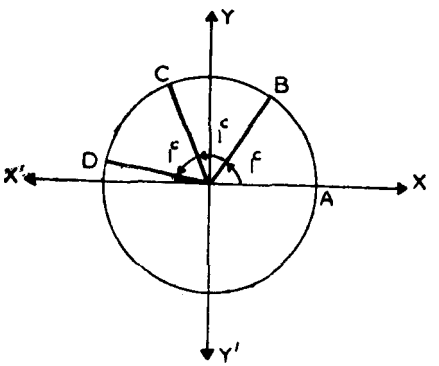


Fig. 5·19

By definition of a radian, $m\angle AOB = m\angle BOC$ $m\angle COD = 1^c$. Now l (arc AC) = l (arc AB) + l (arc BC) = $2r$ and $m\angle AOC = 2^c$. Similarly, l (arc AD) = $3r$ and $m\angle AOD = 3^c$. Hence, when the arc is doubled or tripled, the angle subtended by it at the centre of the circle is also doubled or tripled. Similarly if the arc is halved, the angle subtended by it at the centre is also halved. Hence the angle (measured in radians) subtended at the centre of the circle is proportional to the length of the arc. $\therefore$ If in a circle, an arc of length s subtends an angle θ (measured in radians) at the centre of the circle then $\theta \propto s$.

Hence $\theta = k\,s$ where k is a constant. To evaluate k, we use the fact that when $s = r$, $\theta = 1$. Putting these in $\theta = k\,s$, we get

$$1 = k \cdot r \quad \therefore \quad k = \frac{1}{r}.$$

Hence $\theta = \left(\frac{1}{r}\right) \cdot s$ or $s = r\theta$ as required.

Note: When we use the formula $s = r\theta$, we must take care to see that (i) the angle θ is measured in radians and (ii) lengths of s and r are measured in the same units of length.

§ 5·25. Conversion formula :

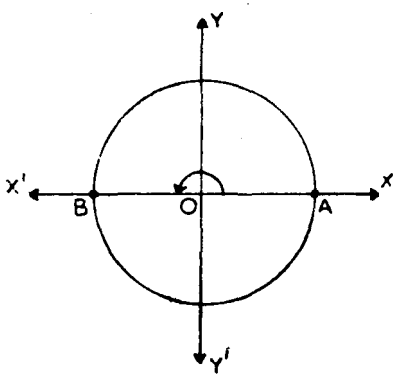


Fig. 5·20

Since we must measure θ in radins while using $s = r\,\theta$, we must have a formula to convert degrees into radians and vice versa. This can be found as follows : Draw a circle with centre O and radius r. Let seg AB be a diameter of this circle. $\therefore$ $m\angle AOB = 180°$. Now let $m\angle AOB = \theta^c$. As the length of arc $AB = \pi r$, by putting $s = \pi r$ in $s = r\theta$ we get

$$\pi r = r\theta \quad \therefore \quad \theta = \pi$$

Hence $\angle AOB = \pi^c$. But $\angle AOB = 180°$. $\therefore$ $\pi^c = 180°$. Hence we get :

$$1^c = \left(\frac{180}{\pi}\right)^\circ \text{ and } 1^\circ = \left(\frac{\pi}{180}\right)^c$$

which can be used to convert radians into degrees and degrees into radians respectively.

§ 5·26. Area of a sector :

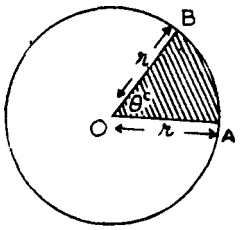


Fig. 5·21

Consider a circle with centre O and radius r. Let A and B be two points on the circumference of the circle. Join OA and OB. Then the shaded portion OAB in fig. 5·21 is called a sector. We assume the following formula for the area of a sector. In fig. 5·21, if $m\angle AOB = \theta$ (measured in radians) and r is the radius of the circle then the area of the sector OAB $\frac{1}{2} r^2 \theta$.

§ 5·27. Examples :

Ex. 1. Convert the following angles into radians :

30°, 45°, 60°, 90°, 120°, 135°, 150°, 270°.

Ans. We know that $1^\circ = \left(\frac{\pi}{180}\right)^c$

$$\therefore \quad 30^\circ = \left(30 \times \frac{\pi}{180}\right)^c = \frac{\pi^c}{6}$$

Similarly we get

$45° = \frac{\pi^c}{4}$; $60° = \frac{\pi^c}{3}$; $90° = \frac{\pi^c}{2}$, $120° = \frac{2\pi^c}{3}$, $135° = \frac{3\pi^c}{4}$,

$150° = \frac{5\pi^c}{6}$; $270° = \frac{3\pi^c}{2}$.

Note : These angles occur frequently in examples and hence may be remembered in terms of circular measures.

Ex. 2. The sum of 2 angles is 1^c and their difference is $20°$. *Find the angles in radians.*

Ans. Let the angles be x and y in circular measures. Also

$$20° = \left(20 \times \frac{\pi}{180}\right)^c = \frac{\pi^c}{9}$$

$$\therefore \quad x + y = 1^c \text{ and } x - y = \frac{\pi^c}{9}.$$

Solving we get $2x = 1 + \frac{\pi}{9} \quad \therefore \quad x = \left(\frac{\pi + 9}{18}\right)^c$

and $2y = 1 - \frac{\pi}{9} \quad \therefore \quad y = \left(\frac{9 - \pi}{18}\right)^c$

Ex. 3. Find the interior angle of a regular pentagon in radians.

Ans. Since the sum of the exterior angles of a polygon is $360°$, each exterior angle of a regular pentagon $= \frac{360°}{5} = 72°$.

$\therefore$ Each interior angle $= 180° - 72° = 108°$

Now $108° = \left(108 \cdot \frac{\pi}{180}\right)^c = \frac{3\pi^c}{5}$

$\therefore$ The interior angle of a regular pentagon is $\frac{3\pi^c}{5}$.

Ex. 4. The angles of a triangle are in the ratio 1 : 3 : 5, Find them in degrees and in radians.

Ans. Let the angles be k, $3k$ and $5k$ in degrees,

$\therefore \quad k + 3k + 5k = 180°$

$\therefore \quad 9k = 180° \qquad \therefore k = 20°$

$\therefore$ The angles in degrees are $20°$, $60°$ and $100°$ .... (I)

Now $20° = \left(20 \cdot \frac{\pi}{180}\right)^c = \frac{\pi^c}{9}$. Similarly $60° = \frac{\pi^c}{3}$

and $100° = \frac{5\pi^c}{9}$.

$\therefore$ The angles in radian are $\frac{\pi^c}{9}$, $\frac{\pi^c}{3}$ and $\frac{5\pi^c}{9}$ .... (II)

Exercise 5·2 :

(1) Convert the following angles into degrees :

$\frac{3\pi^c}{5}$; $\frac{7\pi^c}{3}$; $\frac{4\pi^c}{9}$; $\frac{19\pi^c}{4}$; $\frac{23\pi^c}{5}$

(2) If $x^\circ = \frac{\pi^c}{5}$ and $330^\circ = y^c$ find x and y.

(3) In a triangle ABC, $m\angle A = \frac{\pi^c}{3}$, $m\angle B = 45^\circ$, find $m\angle C$ both in degrees and radians.

(4) If in a $\triangle ABC$, $m\angle A = \frac{5\pi^c}{36}$ and $m\angle B = 120^\circ$, find $m\angle C$ in radians.

(5) In a cyclic quadrilateral ABCD, $m\angle A = 50^\circ$ and $m\angle B = 80^\circ$. Find $m\angle C$ and $m\angle D$ in radians.

(6) Find the circular measure of the interior angle of (i) a regular hexagon, (ii) a regular pentagon.

(7) The angles of a triangle are in the ratio 2 : 3 : 4. Find them in radians as well as in degrees.

(8) The angles of a quadrilateral are in the ratio 2:3:5:8. Find them in radians.

(9) The sum of 2 angles is 200° and their difference is 2^c. Find the angles in radians.

§ 5·28. Examples :

Ex. 1. If in a circle of radius 5 cms., an arc subtends an angle of 72° at the centre, find the length of the arc.

Ans. In the usual notation, we are given that $r = 5$ cms., $\theta = 72^\circ$ and we have to find s.

Now $\theta = 72^\circ = \left[72 \cdot \frac{\pi}{180}\right]^c = \frac{2\pi^c}{5}$

Since $s = r\theta$, $\therefore$ $s = 5 \cdot \left(\frac{2\pi}{5}\right) = 2\pi$ cms.

Ex. 2. In a circle of area 36π sq. cms., an arc subtends an angle of 40° at the centre. Find the length of this arc and the area of the corresponding sector.

Ans. Let the radius of the circle be r cms.

Since Area of the circle $= 36\pi$ sq. cms.

$\therefore$ $\pi r^2 = 36\pi$ $\therefore$ $r = 6$ cms.

Now $\theta = 40° = \left[40 \cdot \frac{\pi}{180}\right]^c = \frac{2\pi^c}{9}$.

$\therefore$ Length of the arc $= s = r\theta = 6 \cdot \left(\frac{2\pi}{9}\right) = \frac{4\pi}{3}$ cms.

and area of the sector $= \frac{1}{2} r^2 \theta = \frac{1}{2} \cdot (6)^2 \cdot \frac{2\pi}{9} = 4\pi$ sq. cms.

Ex. 3. Two concentric circles of radii 4" and 8" are drawn. Two straight lines are drawn through the centre, inclined to each other at 45°. Find the perimeter and the area of the portion lying between the lines and the circles.

Ans. Refer to fig. 5·22. Let O be the common centre of the circles. Let the two lines making an angle of 45° with each other cut the inner circle at A and B and the outer circle at A' and B'.

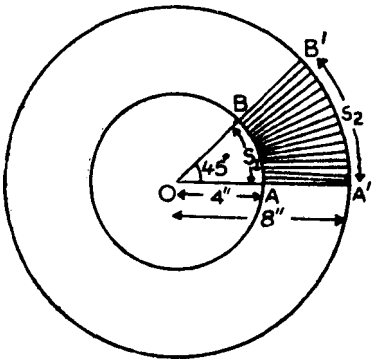


Fig. 5·22

$\therefore$ $l(OA) = 4"$ and $l(OA') = 8"$

$\therefore$ $l(AA') = 4"$.

Similarly $l(BB') = 4"$.

Consider the inner circle Let $l(\text{arc } AB) = s_1$. Since $\theta = 45° = \frac{\pi^c}{4}$, using the formula $s = r\theta$, we get $s_1 = 4\left(\frac{\pi}{4}\right) = \pi"$.

Similarly in the outer circle, if $l(\text{arc } A'B') = s_2$ then $s_2 = 8\left(\frac{\pi}{4}\right) = 2\pi"$.

$\therefore$ Perimeter of the the portion between the lines and the circles $= l(AA') + l(BB') + s_1 + s_2$

$= 4" + 4" + \pi" + 2\pi" = (3\pi + 8)"$.

Also using the formula $\frac{1}{2} r^2\theta$ for the area of a sector, we have,

Area of sector $OAB = \frac{1}{2}(4)^2 \cdot \frac{\pi}{4} = 2\pi$ sq. inches.

and the area of sector $OA'B' = \frac{1}{2}\frac{\pi}{4}(8)^2 = 8\pi$ sq. inches.

Area of portion ABB'A'A

= Area of sector OA'B' – Area of Sector OAB

$= 8\pi - 2\pi = 6\pi$ sq. inches.

Ex. 4. ABC is an equilateral triangle of side 18 cms. A circle is drawn on seg BC as diameter. Find the length of the arc of this circle intercepted within the triangle.

Ans. Refer to fig. 5·23. O is the centre of the circle drawn on seg BC as diameter. Let this circle cut seg AB at M and seg AC at N. ∴ We have to find the length of the arc MN. Join OM and ON.

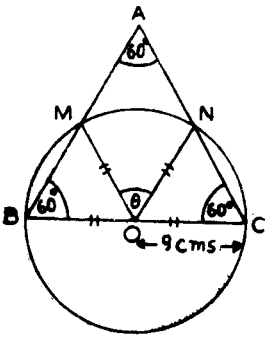


Fig. 5·23

Since $l(OB) = l(OM)$,

$m \angle OMB = m \angle OBM = 60°$.

$\therefore$ $m \angle MOB = 60°$. Similarly $m \angle NOC = 60°$. $\therefore$ $m \angle MON = 60°$.

Hence in the circle, we have

$$r = 9 \text{ cms. and } \theta = 60° = \frac{\pi^c}{3}$$

$$\therefore \quad l(\text{arc MN}) = 9\left(\frac{\pi}{3}\right) = 3\pi \text{ cms.}$$

Ex. 5. O is the centre of a circle of radius 6 cms. seg OP and seg OQ are two radii of the circle such that $m \angle POQ = 30°$. Find (i) the length of arc PQ and (ii) the area between the chord PQ and the arc PQ.

Ans. Let O be the centre of the circle with radius 6 cms. Draw two radii OP and OQ such that $\angle POQ = 30°$. Join PQ. Draw QM perpendicular to seg OP meeting seg OP at M.

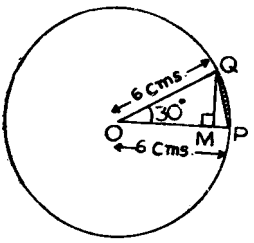


Fig. 5·24

We have to find the length of arc PQ and the area between arc PQ and chord PQ which has been shaded in figure 5·24.

Now $l(\text{arc PQ}) = r\theta$

Here $r = 6$ cm. and $\theta = 30° = \frac{\pi^c}{6}$

$$\therefore l(\text{arc PQ}) = 6 \cdot \frac{\pi}{6} = \pi \text{ cms.} \quad \ldots\ldots \quad (i)$$

Obviously, area of the shaded portion = area of sector OPQ − area of triangle OPQ.

Now area of sector OPQ $= \frac{1}{2}r^2\theta = \frac{1}{2}(6)^2 \cdot \left(\frac{\pi}{6}\right) = 3\pi$ sq. cms.

and the area of triangle OPQ $= \frac{1}{2}$ OP·QM

$= \frac{1}{2}(6)(3)$, [Since $m\angle MOQ = 30°$, $\therefore$ $l(QM) = \frac{1}{2}\cdot l(OQ) = 3$].

$= 9$ sq. cms.

$\therefore$ Area of shaded portion $= 3\pi - 9 = 3(\pi - 3)$ sq. cms..... (ii)

Exercise 5.3 :

(1) Find the length of the arc AB of a circle of radius 6", given that the arc subtends an angle of 200° at the centre of the circle. Also find the area of the corresponding sector.

(2) The radius of a circle is 5 cms. Find the length of an arc of this circle which cuts off a chord of length equal to the radius.

(3) The area of a circle is 36π sq. inches. Find the area of a sector of this circle which contains an angle of $\frac{5\pi^c}{12}$

(4) The area of a circle is 81π sq. cms. Find the length of its arc subtending an angle of 30° at the centre. Also find the area of the corresponding sector.

(5) AB is the arc of a circle subtending an angle of 144° at the centre. If the radius of the circle is 10", find the length of arc AB and area of the corresponding sector.

(6) An arc of length 3π cms. of a sector of a circle subtends an angle of 60° at the centre of the circle. Find the radius of the circle, the area of the sector and the area of the circle.

(7) The chords AP and AQ of a circle of radius 6 cms. are at distances 3 cms. and $3\sqrt{2}$ cms. respectively from the centre of the circle. If O be the centre of the circle find the perimeter and the area of the sector POQ bounded by the minor arc PQ.

(8) A hollow right circular cone is made from a piece of paper in the form of a sector of a circle of radius 50 cms. and angle 216° by joining the straight edges of the sector Find the height of the cone.

(9) Two concentric circles of radii 2" and 3" are drawn and two straight lines are drawn through the centre inclined at an angle of 30°. Show that the perimeter and the area of the portion between the straight lines and the two circles are $\left(\frac{5\pi+12}{6}\right)$" and $\left(\frac{5\pi}{12}\right)$ sq. inches respectively.

(10) Find the ratio of the radii of two circles, at the centres of which, arcs of the same length subtend angles of 30° and 45°

(11) Two circles, each of radius a, intersect each other. The distance between their centres is $\sqrt{2}\,a$. Find the area of the portion common to the two circles.

(12) ABC is an equilateral triangle with side 8 cms. A circle is drawn on BC as diameter. Show that the length of the arc of the circle included within the triangle is $\frac{4\pi}{3}$ cms.

(13) In a triangle ABC the angles B and C are each 55°. A circle is drawn on BC as diameter. Find the length of the arc of this circle intercepted within the triangle if $l(BC) = 18$ cms.

§ 5·3. Trigonometric Ratios :

§ 5·31. Trigonometric Ratios of Acute Angles :

You have already studied the trigonometric ratios named sine, cosine and tangent of an acute angle θ in a right triangle as

$$\sin\theta = \frac{\text{length of side opposite to } \theta}{\text{length of hypotenuse}}$$

$$\cos\theta = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$$

$$\tan\theta = \frac{\text{length of side opposite to } \theta}{\text{length of side adjacent to } \theta}$$

and with the help of these you have also found the values of the ratios for angles of measures 30°, 45° and 60° which are as follows :

Trigonometric Ratio	Measure of the angle		
	30°	45°	60°
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Using these values you can solve the following examples.

Exercise 5·4 :

Prove the following :

(1) $\sin 30° \cdot \cos 60° + \cos 30° \cdot \sin 60° = 1$

(2) $2\tan^2 45° + 3\tan^2 60° = 11$

(3) $\tan 45° - 2\sin^2 30° - \cos 60° = 0$

(4) $\dfrac{1-\tan^2 30°}{1+\tan^2 30°} - \cos 60° = 0$

(5) $4\sin^2 30° + 16\cos^2 60° = 5$

(6) $16\sin^3 30° + 8\cos^3 60° = 3$

(7) $3\sin 30° - 4\sin^3 30° = 1$

(8) $4\cos^3 30° - 3\cos 30° = 0$

§ 5·32. With the help of these definitions we can find $\sin\theta, \cos\theta, \tan\theta$ only when θ is an acute angle, i. e. when a right triangle, with θ as one of the two acute angles, can be drawn.

We have now seen that there exist angles of measures 135°, 315°, 360°, 390°, 780° etc., and for finding the trigonometric ratios of such angles we cannot find any right triangle to use the definitions mentioned above. Hence we now define the trigonometric ratios $\sin\theta$, $\cos\theta$, $\tan\theta$ and some others, in general, for any angle θ which can be used to find trigonometric ratios of angles such as 135°, 315°, 360°, 390°, 780° etc., and which also can be used for finding trigonometric ratios of acute angles also such as 30°, 45°, 60°.

§ 5·33. **Trigonometric Ratios of any angle :**

Let θ be an angle in standard position. Let $P(x, y)$ be any point on the terminal side of angle θ with polar distance $r > 0$, then we define :

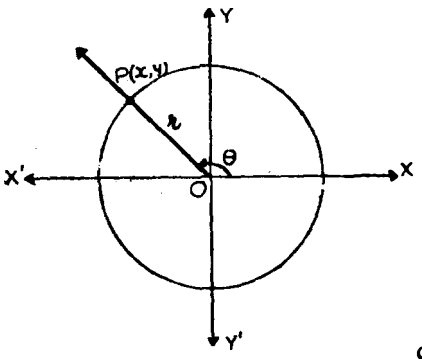


Fig. 5·25

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\cot\theta = \frac{x}{y} \quad (y \neq 0)$$

$$\sec\theta = \frac{r}{x} \quad (x \neq 0)$$

$$\operatorname{cosec}\theta = \frac{r}{y} \quad (y \neq 0)$$

sin, cos, tan, cot, sec, cosec are the abbreviations of sine, cosine, tangent, cotangent, secant and cosecant ratios of an angle θ, e. g. secant of an angle $\theta = \frac{r}{x}$ $(x \neq 0)$ is written in short as $\sec\theta = \frac{r}{x}$ $(x \neq 0)$.

The line segment OP is called the *radius vector* of the point P and the length of OP which is 'non-negative' is called the *polar distance* of P.

The coordinates x and y of P can be positive, negative or zero depending on the position of the terminal line of angle θ, but the polar distance r cannot be negative and further, since $P(x, y) \neq (0, 0)$, $r \neq 0$.

Given an angle θ, $\sin\theta$, $\cos\theta$, $\tan\theta$ .... etc. have unique values whenever they exist. These ratios, therafore, depend on θ only and not on r, i.e. for a given θ, $\sin\theta$, $\cos\theta$ etc. remain the same whatever may the value of r, i. e., these trigonometric ratios are *independent of r*. This is proved as follows,

§ 5·34. Theorem 5·3:

"The trigonometric ratios $\sin\theta$, $\cos\theta$, ... **etc. are independent of** r**".**

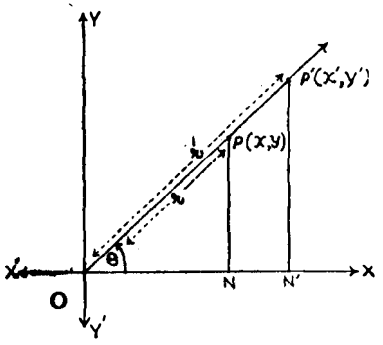


Fig. 5·26

Let θ be the angle in the standard position with x axis as the initial side and OB as the terminal side of the angle. Let $P(x, y)$ be any point on the terminal side OB such that the line segment OP is of length r (i. e. the polar distance of P is r).

$$\therefore \sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x}, \text{ etc.} \quad \ldots\ (i)$$

Now let $P'(x', y')$ be any point of the terminal side OB such that the line segment OP' is of length r' (i. e. the polar distance of P' is r').

Hence $\sin\theta=\frac{y'}{r'}$, $\cos\theta=\frac{x'}{r'}$, $\tan\theta=\frac{y'}{x'}$ etc. . . . (ii)

We have now to show that $\sin\theta$, $\cos\theta$, $\tan\theta$, etc. as defined in (i) and (ii) give the unique value i. e.

$$\frac{y}{r}=\frac{y'}{r'},\ \frac{x}{r}=\frac{x'}{r'},\ \frac{y}{x}=\frac{y'}{x'}\ \ldots\ldots \text{ etc.}$$

Now triangles OPN and OP'N' are similar.

$$\therefore\quad \frac{l(OP)}{l(OP')}=\frac{l(ON)}{l(ON')}=\frac{l(PN)}{l(PN')} \qquad \ldots\ (iii)$$

But $l(OP)=r$, $l(OP')=r'$, $l(ON)=x$, $l(ON')=x'$, $l(PN)=y$, $l(PN')=y'$.

$\therefore$ From (iii) $\frac{r}{r'}=\frac{y}{y'}=\frac{x}{x'}$

$$\therefore\quad \frac{y}{r}=\frac{y'}{r'},\ \frac{x}{r}=\frac{x'}{r'},\ \frac{y}{x}=\frac{y'}{x'} \text{ etc.}$$

Hence, whatever may be the length of the radius, the trigonometric ratios $\sin\theta$, $\cos\theta$, $\tan\theta$ etc. have unique values.

Note : When we take $r=1$ we get

$$\sin\theta=\frac{y}{r}=\frac{y}{1}=y \text{ and } \cos\theta=\frac{x}{r}=\frac{x}{1}=x$$

i. e. x coordinate of P is $\cos\theta$ and y coordinate of P is $\sin\theta$, as in § 4.3 on circular functions.

5.35. Signs of Triogonometric Ratios :

The coordinates x and y of any point P on the terminal side of an angle may be positive or negative depending on the quadrant in which the terminal side lies but r is always positive. Hence the signs of the trigonometric ratios will depend on the signs of x and y.

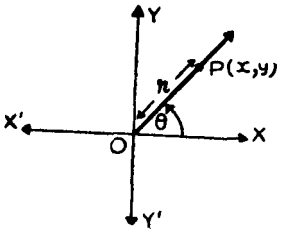


Fig. 5·27

Quadrant I : If the terminal side of angle θ (in standard position) is in the first quadrant, then both x and y coordinates of P are positive. $\therefore$ x, y and r are all positive. Hence all the six trigonometric ratios are positive.

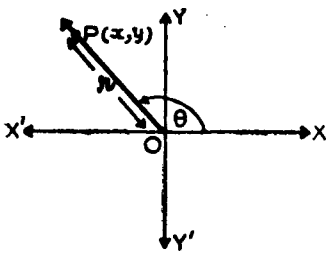


Fig. 5·28

Quadrant II : If P (x, y) is in the second quadrant, x coordinate is negative and y coordinate is positive. Further r is always positive. Since only x is negative $\sin\theta = \frac{y}{r}$ and $\operatorname{cosec}\theta = \frac{r}{y}$ are both positive whereas $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$ and $\sec\theta = \frac{r}{x}$ are all negative.

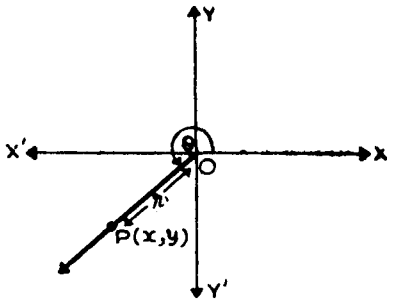


Fig. 5·29

Quadrant III. If P (x, y) is in the third quadrant, both x and y coordinates are negative and only r is positive. Hence, $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\sec\theta = \frac{r}{x}$, $\operatorname{cosec}\theta = \frac{r}{y}$ are all negative, whereas $\tan\theta = \frac{y}{x}$ and $\cot\theta = \frac{x}{y}$ are both positive.

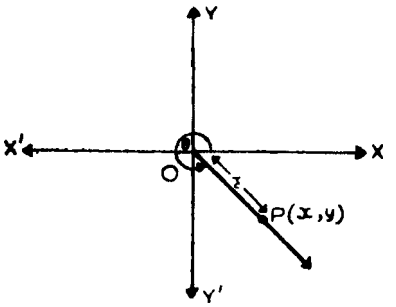


Fig. 5·30

Quadrant IV. If P(x,y) is in the fourth quadrant, x coordinate is positive but y coordinate is negative. Further r is always positive. ∴ Since only y is negative, $\cos\theta = \frac{x}{r}$ and $\sec\theta = \frac{r}{x}$ are both positive, whereas $\sin\theta = \frac{y}{r}$, $\operatorname{cosec}\theta = \frac{r}{y}$, $\tan\theta = \frac{y}{x}$ and $\cot\theta = \frac{x}{y}$ are all negative.

§ 5·4. Trigonometric functions :

We have seen that for any given θ, each of the trigonometric

ratios is uniquely determined, Hence with the help of the relations between trigonometric ratios, we can define the trigonometric functions; sine function, cosine function, tangent function, cotangent function, secant function and cosecant function as follows :

Defintions : If θ is an angle in the standard position and if $P(x, y)$, where $[(x, y) \neq (0, 0)]$ is any point on the terminal side of the angle θ then we define the trigonometric functions as follows :

(1) Sine: $\theta \to \sin\theta$ is defined as:

$$\text{Sine} \equiv \left\{ (\theta, \sin\theta) \mid \theta \in R, \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \right\}$$

Domain of sine function is R and range is $-1 \leqslant \sin\theta \leqslant +1$.

(2) Cosine: $\theta \to \cos\theta$ is defined as:

$$\text{Cosine} \equiv \left\{ (\theta, \cos\theta) \mid \theta \in R, \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \right\}$$

Domain of cosine function is R and range is $-1 \leqslant \cos\theta \leqslant +1$

(3) Tangent: $\theta \to \tan\theta$ is define as:

$$\text{Tangent} \equiv \left\{ (\theta, \tan\theta) \mid \theta \in R, \theta \neq (2n-1)\frac{\pi}{2}, n \in I; \tan\theta = \frac{y}{x} \right\}$$

Domain of tangent function is

$$\left\{ \theta \mid \theta \in R, \theta \neq (2n-1)\frac{\pi}{2}, n \in I \right\} \text{ and range is R.}$$

(4) Cotangent: $\theta \to \cot\theta$ is defined as:

$$\text{Cotangent} \equiv \{ (\theta, \cot\theta) \mid \theta \in R, \theta \neq n\pi, n \in I; \cot\theta = \frac{x}{y} \}$$

Domain of cotangent function is $\{\theta \mid \theta \in R, \theta \neq n\pi, n \in I\}$ and range is R.

(5) Secant: $\theta \to \sec\theta$ is defined as:

$$\text{Secant} \equiv \left\{ \begin{array}{l} (\theta, \sec\theta) \ \theta \in R, \theta \neq (2n-1)\frac{\pi}{2}, n \in I, \\ \qquad\qquad \sec\theta = \frac{\sqrt{x^2+y^2}}{x} \end{array} \right\}$$

Domain of secant function is $\left\{\theta \mid \theta \in R, \theta \neq (2n-1)\frac{\pi}{2}, n \in I\right\}$ and range is $\{\sec\theta \mid \sec\theta \leqslant -1 \text{ or } \sec\theta \geqslant 1\}$.

(6) Cosecant : $\theta \to \text{cosec}\,\theta$ is defined as :

$$\text{Cosecant} \equiv \left\{ (\theta, \text{cosec}\,\theta) \mid \theta \neq n\pi, n \in I, \text{cosec}\,\theta = \frac{\sqrt{x^2+y^2}}{y} \right\}$$

Domain of cosecant function is $\{\theta \mid \theta \in R, \theta \neq n\pi, n \in I\}$ and range is $\{\text{cosec } \theta \mid \text{cosec } \theta \leqslant -1 \text{ or cosec } \theta \geqslant 1\}$.

These functions are called Trigonometric functions. The condition $(x, y) \neq (0, 0)$ means that the point P (x, y) is any point other than origin $(0, 0)$ on the terminal side. Hence x and y cannot be both zero at the same time. But one of them can be zero. Hence in the tangent function, the domain contains the condition $\theta \neq (2n-1)\frac{\pi}{2}$, $n \in I$ so that θ is not an odd multiple of $\frac{\pi}{2}$ which ensures that the terminal side does not coincide with y axis. Then obviously $x \neq 0$ and $\tan \theta = \frac{y}{x}$ will exist. The same is true for secant function.

Similarly $\theta \neq n\pi$, $n \in I$, is a condition in the domain of cotangent and cosecant functions which will ensure that the terminal side does not coincide with x axis and hence $y \neq 0$,

$\therefore \cot \theta = \frac{x}{y}$ and $\text{cosec } \theta = \frac{\sqrt{x^2+y^2}}{y}$ will both exist.

Further $\sqrt{x^2+y^2}$ is to be taken as the positive square root of x^2+y^2.

§ 5·41. Angles not in standard position :

When an angle α is not in standard position, we can take another angle α', in standard position, congruent to the given angle α, as shown in fig. 5·31. Since the trigonometric functions of α and α' are same, we can get trigonometric functions of α by finding the trigonometric functions of α'.

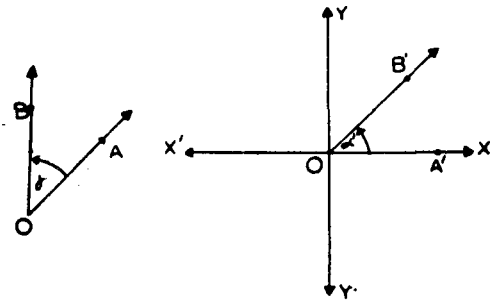


Fig. 5·31

Ex. 1. If $(-3, 4)$ is any point on the terminal side of angle θ, find the corresponding element in the range of each of the six trigonometric functions.

Ans. $(x, y) \equiv (-3, 4)$ $\therefore \quad x = -3, \quad y = 4.$

$\therefore \quad \sqrt{x^2+y^2} = +\sqrt{(-3)^2+(4)^2} = +\sqrt{9+16} = +\sqrt{25} = 5$

$\therefore \quad x = -3, \quad y = 4, \quad \sqrt{x^2+y^2} = 5$

$$\therefore \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{4}{5}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{-3}{5} = -\frac{3}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$

$$\cot\theta = \frac{x}{y} = \frac{-3}{4} = -\frac{3}{4}$$

$$\sec\theta = \frac{\sqrt{x^2+y^2}}{x} = \frac{5}{-3} = -\frac{5}{3}$$

$$\operatorname{cosec}\theta = \frac{\sqrt{x^2+y^2}}{y} = \frac{5}{4}.$$

Exercise 5·5 :

Find the elements in the range of each of the six trigonometric functions, which corresponds to θ, if the terminal side of angle θ contains the point :

(a) $(-3, -4)$ (b) $(-7, 24)$ (c) $(5, -12)$
(d) $(-7, -24)$ (e) $(-5, -12)$ (f) $(1, 1)$
(g) $(6, 8)$ (h) $(-1, -1)$ (i) (c, c).

§ 5·42. Relation between Circular and Trigonometric Functions :

We have defined the trigonometric functions of angle θ in terms of x and y, where x and y are the x and y coordinates of *any* point P on the terminal side of the angle θ. Since P can be any point on OB, we may select P such that P is the point of intersection of line OB with the unit circle.

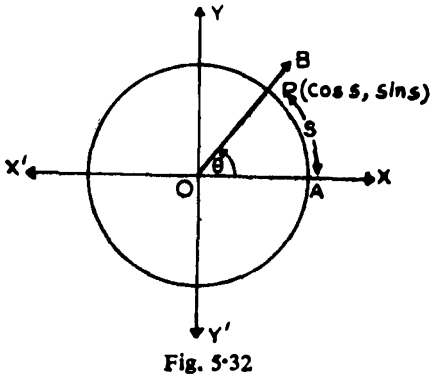


Fig. 5·32

$\therefore \quad l(\mathrm{OP}) = 1,$

i. e. $\sqrt{x^2+y^2} = 1$

Hence $\sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{1} = y$ $\quad \therefore\ y = \sin\theta$

and $\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{1} = x,$ $\quad \therefore\ x = \cos\theta$

$\therefore$ P (x, y) = P $(\cos\theta, \sin\theta)$ .. .. (i)

But P, a point on the unit circle, has coordinates $(\cos s, \sin s)$ where s is the length of arc AP intercepted by θ on the unit circle.

$\therefore$ P (x, y) = P $(\cos s, \sin s)$ .. .. (ii)

From (i) and (ii), we get $\cos\theta = \cos s$ and $\sin\theta = \sin s$.

Thus, the elements in the ranges of the trignometric functions $\sin\theta$, $\cos\theta$ are the same as the corresponding elements in the ranges of the circular functions $\sin s$, $\cos s$ where s is the length of the arc intercepted by θ on the unit circle.

Thus, the elements in the ranges of each of trignometric functions and circular functions are real numbers, but the domain of trigonometric functions are *angles* whereas the elements s in the domain of the circular functions are real numbers.

If the angle θ is such that the length of the arc intercepted by it on the unit circle is $\frac{\pi}{4}$, then the corresponding element in the range of circular function $\cos s$ is $\cos\frac{\pi}{4}$. The measure of the angle θ now will be $\frac{\pi}{4}$ radians or 45° and the corresponding element in the range of Trigonometric function $\cos x$ is $\cos\left(\frac{\pi^c}{4}\right)$ or $\cos 45°$.

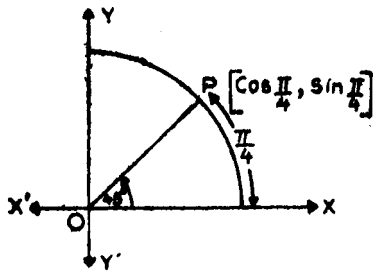


Fig. 5·3 3

$\therefore$ $\cos 45° = \cos\frac{\pi}{4}$ and $\cos\left(\frac{\pi^c}{4}\right) = \cos\frac{\pi}{4}$ where 45° or $\frac{\pi}{4}$ radians is the measure of the angle whereas in $\cos\frac{\pi}{4}$ on R. H. S. for circular function, $\frac{\pi}{4}$ is the length of the arc AP. $\therefore$ $\cos 45°$ or $\cos\left(\frac{\pi^c}{4}\right)$ is an element in the range of circular function.

If s is the length of the arc intercepted by θ on the unit circle and if we denote trignometric functions by T and circular functions by C then $T(\theta) = C(s)$. The elements θ in the domain of T are angles, and the elements s in the domain of C are real numbers. Ranges, however, for both functions T and C are real numbers.

§ 5·43. Trigonometric Functions of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ etc.

For finding $\sin\theta$, $\cos\theta$, $\tan\theta$ etc. for any θ, we can take a point $P(x, y)$ on the terminal side of the angle θ and use the definitions: $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$ etc. where $r = +\sqrt{x^2+y^2}$.

(1) Trigonometric of 0 and $\frac{\pi}{2}$:

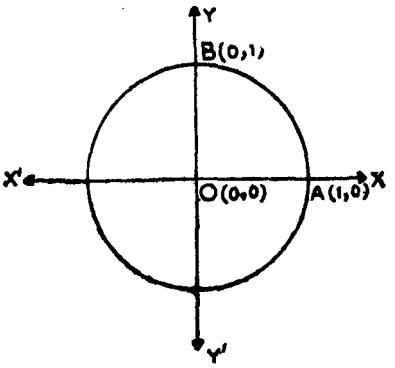


Fig. 5·34

Draw a unit circle with centre O. Let this circle intersect the terminal sides of angles 0 and $\frac{\pi}{2}$ at A and B respectively.

$A \equiv (1, 0)$ and $B \equiv (0, 1)$ and r in each case is 1.

$$\therefore \sin 0 = \frac{y \text{ coordinate of A}}{r} = \frac{0}{1} = 0$$

$$\cos 0 = \frac{x \text{ coordinate of A}}{r} = \frac{1}{1} = 1$$

$$\tan 0 = \frac{y \text{ coordinate of A}}{x \text{ coordinate of A}} = \frac{0}{1} = 0$$

$$\sin \frac{\pi}{2} = \frac{y \text{ coordinate of P}}{r} = \frac{1}{1} = 1$$

$$\cos \frac{\pi}{2} = \frac{x \text{ coordinate of P}}{r} = \frac{0}{1} = 0$$

$\tan \frac{\pi}{2}$ is not defined because x coordinate of P is 0.

Similarly using $\sec\theta = \frac{r}{x}$, $\operatorname{cosec}\theta = \frac{r}{y}$ and $\cot\theta = \frac{x}{y}$ we can find other trigonometric ratios of angles 0 and $\frac{\pi}{2}$, whenever they exist.

(2) Trigonometric ratios of $\frac{\pi}{6}$:

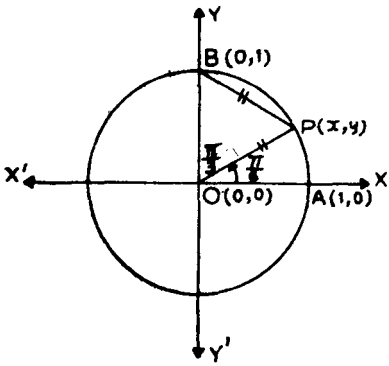


Fig 5·35

Draw a unit circle with centre O (0, 0). Let A ≡ (1, 0). Let P (x, y) be the point of intersection of the terminal side of angle $\frac{\pi}{6}$, in standard position with the unit circle. Let B ≡ (0, 1).

∴ measure of the angle POB is $\frac{\pi}{3}$.

∴ OPB is an equilateral triangle.

∴ $l(OP) = l(PB)$

$$\therefore \quad [l(OP)]^2 = [l(PB)]^2$$

$$\therefore \quad (x-0)^2 + (y-1)^2 = x^2 + y^2$$

$$\therefore \quad x^2 + y^2 - 2y + 1 = x^2 + y^2$$

$$\therefore \quad -2y + 1 = 0 \qquad \therefore \quad 2y = 1, \quad y = \frac{1}{2}$$

But $x^2 + y^2 = 1$ $\quad \therefore \quad x^2 = 1 - y^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$$\therefore \quad x = +\frac{\sqrt{3}}{2} \quad (\because \text{ P is in the first quadrant})$$

$$\therefore \quad x = \frac{\sqrt{3}}{2}, \; y = \frac{1}{2}, \; r = 1$$

$$\therefore \quad \sin\frac{\pi}{6} = \frac{y \text{ coordinate of P}}{r} = \frac{(1/2)}{1} = \frac{1}{2},$$

$$\therefore \quad \cos\frac{\pi}{6} = \frac{x \text{ coordinate of P}}{r} = \frac{(\sqrt{3}/2)}{1} = \frac{\sqrt{3}}{2},$$

$$\therefore \quad \tan\frac{\pi}{6} = \frac{y \text{ coordinate of P}}{x \text{ coordinate of P}} = \frac{(1/2)}{(\sqrt{3}/2)} = \frac{1}{\sqrt{3}}.$$

(3) Trigonometric ratios of $\frac{\pi}{4}$:

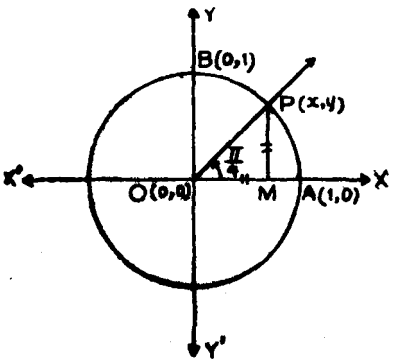


Fig. 5·36

Draw a unit circle with centre O (0, 0). Let A ≡ (1, 0). Let P (x, y) be the point of intersection of the terminal side of the angle $\frac{\pi}{4}$ (in standard position) with the unit circle. Draw PM perpendicular to x axis.

$\therefore x = l(\text{OM}),\ y = l(\text{PM})$.

Now in right angled triangle OPM, $m\angle\text{POM} = \frac{\pi}{4}$.

$\therefore l(\text{OM}) = l(\text{PM}) \quad \therefore x = y$.

But $x^2 + y^2 = 1 \quad \therefore x^2 + x^2 = 1,\ 2x^2 = 1$

$\therefore x^2 = \frac{1}{2},\ x = +\frac{1}{\sqrt{2}} \quad \because$ P (x, y) is in the first quadrant.

$\therefore x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$ ($\because x = y$) and $r = 1$

$$\therefore \sin\frac{\pi}{4} = \frac{y \text{ coordinate of P}}{r} = \frac{(1/\sqrt{2})}{1} = \frac{1}{\sqrt{2}},$$

$$\cos\frac{\pi}{4} = \frac{x \text{ coordinate of P}}{r} = \frac{(1/\sqrt{2})}{1} = \frac{1}{\sqrt{2}},$$

$$\tan\frac{\pi}{4} = \frac{y \text{ coordinate of P}}{x \text{ coordinate of P}} = \frac{(1/\sqrt{2})}{(1/\sqrt{2})} = 1.$$

(4) Trigonometric Ratios of $\frac{\pi}{3}$

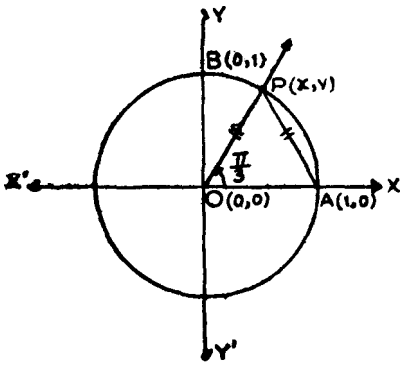


Fig. 5·37

Draw a unit circle with centre O (0, 0). Let A ≡ (1, 0). Let P (x, y) be the point of intersection of the terminal side of angle $\frac{\pi}{3}$ (in standard position), with the unit circle. Since $l(\text{OA}) = l(\text{OP}) = 1$ and $m\angle\text{AOP} = \frac{\pi}{3}$, the triangle OPA is an equilateral triangle.

$\therefore\ l(PA)=l(OP)\ \ \therefore\ [l(PA)]^2=[l(OP)]^2$

$\therefore\ (x-1)^2+(y-0)^2=x^2+y^2$

$\therefore\ x^2-2x+1+y^2=x^2+y^2$

$\therefore\ -2x+1=0\ \ \therefore\ 2x=1\ \ \therefore\ x=\frac{1}{2}$

$\therefore\ y^2=1-x^2=1-(\frac{1}{2})^2=1-\frac{1}{4}=\frac{3}{4}$

$\therefore\ y=+\frac{\sqrt{3}}{2}\ \ \because$ P is in first quadrant.

$\therefore\ x=\frac{1}{2},\ y=\frac{\sqrt{3}}{2}$ and $r=1$

$$\therefore\ \sin\frac{\pi}{3}=\frac{y\text{ coordinate of P}}{r}=\frac{\sqrt{3}/2}{1}=\frac{\sqrt{3}}{2},$$

$$\cos\frac{\pi}{3}=\frac{x\text{ coordinate of P}}{r}=\frac{1/2}{1}=\frac{1}{2},$$

$$\tan\frac{\pi}{3}=\frac{y\text{ coordinate of P}}{x\text{ coordinate of P}}=\frac{\sqrt{3}/2}{1/2}=\sqrt{3}.$$

(5) Trigonometric ratios of $\frac{3\pi}{4}$:

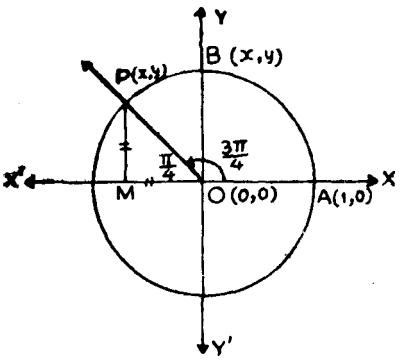


Fig. 5·38

Draw a unit circle with centre O (0, 0). Let P (x, y) be the point of intersection of the terminal side of angle $\frac{3\pi}{4}$, in standard position, with the unit circle.

Draw PM perpendicular to x axis.

In $\triangle$ OPM,

$m\angle POM=\pi-\frac{3\pi}{4}=\frac{\pi}{4}.$

$\therefore\ l(OM)=l(PM).$

But $[l(OM)]^2+[l(PM)]^2=1$

$\therefore\ [l(OM)]^2+[l(OM)]^2=1$

$\therefore\ 2[l(OM)]^2=1\ \ \therefore\ [l(OM)]^2=\frac{1}{2}$

$\therefore\ l(OM)=\frac{1}{\sqrt{2}}\ \ \therefore\ l(PM)=\frac{1}{\sqrt{2}}$

$\therefore\ x=-l(OM)=-\frac{1}{\sqrt{2}}$ and $y=l(PM)=\frac{1}{\sqrt{2}}$

$\because$ P lies in 2nd quadrant

$\therefore$ $P \equiv \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Hence $x = -\frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$ and $r = 1$

$\therefore$ $\sin\frac{3\pi}{4} = \frac{y \text{ coordinate of P}}{r} = \frac{1/\sqrt{2}}{1} = \frac{1}{\sqrt{2}}$.

$\cos\frac{3\pi}{4} = \frac{x \text{ coordinate of P}}{r} = \frac{-1/\sqrt{2}}{1} = \frac{-1}{\sqrt{2}}$.

$\tan\frac{3\pi}{4} = \frac{y \text{ coordinate of P}}{x \text{ coordinate of P}} = \frac{1/\sqrt{2}}{-1/\sqrt{2}} = -1$.

(6) **Trigonometric ratios of $\frac{2\pi}{3}$:**

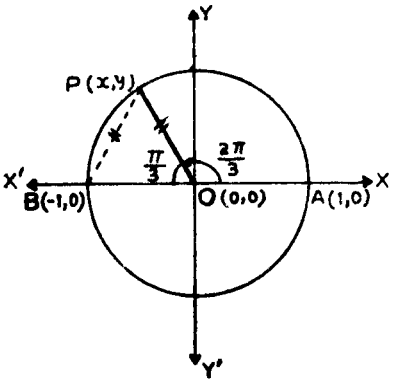


Fig. 5·39

Draw a unit circle with centre O (0, 0). Let P (x, y) be the point of intersection of the terminal side of angle $\frac{2\pi}{3}$, in standard position, with the unit circle. Let B $\equiv (-1, 0)$

$\therefore$ $\angle POB = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

$\therefore$ $\triangle$ OPB is an equilateral triangle.

$\therefore$ $l(PB) = l(OP)$

$\therefore$ $[l(PB)]^2 = [l(OP)]^2$

$\therefore$ $(x+1)^2 + (y-0)^2 = x^2 + y^2$

$\therefore$ $x^2 + 2x + 1 + y^2 = x^2 + y^2$

$\therefore$ $2x + 1 = 0$ $\therefore$ $2x = -1$ $\therefore$ $x = -\frac{1}{2}$.

$\therefore$ $y^2 = 1 - x^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ $y = \frac{+\sqrt{3}}{2}$ $\because$ P is in the second quadrant.

$\therefore$ $x = -\frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$ and $r = 1$

$$\therefore \quad \sin \frac{2\pi}{3} = \frac{y \text{ coordinate of P}}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2},$$

$$\cos \frac{2\pi}{3} = \frac{x \text{ coordinate of P}}{r} = \frac{-1/2}{1} = -\frac{1}{2},$$

$$\tan \frac{2\pi}{3} = \frac{y \text{ coordinate of P}}{x \text{ coordinate of P}} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}.$$

Note : Similarly trigonometric ratios of other angles like $\frac{4\pi}{3}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, etc. can be found.

Exercise 5·6 :

A. Find the following from the definition :

(1) $\sin \frac{5\pi}{4}$ (2) $\sec \frac{3\pi}{4}$ (3) $\cot \frac{7\pi}{4}$

(4) $\cot \frac{4\pi}{3}$ (5) $\operatorname{cosec} \frac{5\pi}{3}$ (6) $\operatorname{cosec} \frac{5\pi}{6}$

(7) $\cot \frac{7\pi}{6}$ (8) $\sin \frac{11\pi}{6}$

B. Find the following from the definition :

(1) $\sin n\pi$, (n is even, $n \in$ I).

(2) $\cos n\pi$, (n is odd, $n \in$ I).

(3) $\cos 2n\pi$, ($n \in$ I).

(4) $\sin (2n+1)\pi$ ($n \in$ I).

§ 5·5. Identities :

The trigonometrical ratios $\sin\theta$, $\cos\theta$, $\tan\theta$ etc. are inter-related in many ways. Such relationships are called *identities* because they hold good for every permissible value of the variables involved. Two such inter-relations are :

(1) For all θ such that $\cos\theta \neq 0$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$.

Proof : As $\cos\theta \neq 0$, $\cos\theta = \frac{x}{r} \neq 0$ $\quad \therefore\ x \neq 0.$

Since $x \neq 0$, $\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}$. $\quad (\because\ r \neq 0).$

$\because\ x \neq 0$, $\sec\theta = \frac{r}{x} = \frac{1}{x/r} = \frac{1}{\cos\theta}$.

(2) For all θ such that $\sin\theta \neq 0$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$.

Proof: As $\sin\theta \neq 0$, $\sin\theta = \frac{y}{r} \neq 0 \quad \therefore \quad y \neq 0$.

Since $y \neq 0$, $\cot\theta = \frac{x}{y} = \frac{x/r}{y/r} = \frac{\cos\theta}{\sin\theta} \quad (\because r \neq 0)$

$\because \ y \neq 0$, $\operatorname{cosec}\theta = \frac{r}{y} = \frac{1}{y/r} = \frac{1}{\sin\theta}$.

Theorem 5·4:

(1) $\sin^2\theta + \cos^2\theta = 1$ for all θ,

(2) $1 + \tan^2\theta = \sec^2\theta$ for all θ where $\cos\theta \neq 0$,

(3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ for all θ where $\sin\theta \neq 0$.

Proof: Let the angle θ be in the standard position.

Let P (x, y) be any point on the terminal side of angle θ such that $l(OP) = r \ (r > 0)$

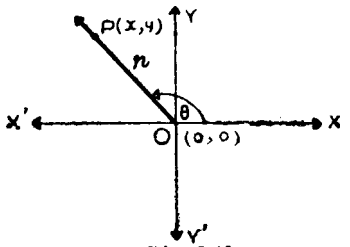


Fig. 5·40

$\therefore \quad \sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}$

Now $\because \ l(OP) = r$

$\therefore \quad \sqrt{(x-0)^2 + (y-0)^2} = r$

$\therefore \quad x^2 + y^2 = r^2 \quad \ldots \quad (i)$

Divide by r^2.

$\therefore \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad \therefore \quad \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$

$\therefore \quad (\cos\theta)^2 + (\sin\theta)^2 = 1.$

We write $(\sin\theta)^2$ as $\sin^2\theta$ and $(\cos\theta)^2$ as $\cos^2\theta$.

$\therefore \quad \sin^2\theta + \cos^2\theta = 1$ for any $\theta \quad \ldots\ldots \quad (1)$

From (i), $x^2 + y^2 = r^2$

If $\cos\theta \neq 0$, i. e. $\frac{x}{r} \neq 0$. i. e. $x \neq 0$, then dividing by x^2, we get

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2 \quad \therefore 1 + (\tan\theta)^2 = (\sec\theta)^2$$

$\therefore \quad 1+\tan^2\theta = \sec^2\theta$ for any θ, where $\cos\theta \neq 0$ .... (2)

From (i), $x^2+y^2=r^2$

If $\sin\theta \neq 0$, i. e. $\frac{y}{r} \neq 0$, i. e. $y \neq 0$, then dividing by y^2, we have: $\frac{x^2}{y^2}+1=\frac{r^2}{y^2}$. $\therefore \left(\frac{x}{y}\right)^2+1=\left(\frac{r}{y}\right)^2$

$\therefore \quad (\cot\theta)^2+1=(\operatorname{cosec}\theta)^2$

$\therefore \quad 1+\cot^2\theta = \operatorname{cosec}^2\theta$ for any θ, where $\sin\theta \neq 0$...(3)

(1), (2) and (3) give the required results.

§ 5·51. Examples :

Ex. 1. If $\sin\theta=\frac{1}{2}$ and θ lies in the second quadrant, find $\cos\theta$ and $\tan\theta$.

Ans. We have $\sin\theta=\frac{1}{2}$

$\cos^2\theta=1-\sin^2\theta=1-(\frac{1}{2})^2=1-\frac{1}{4}=\frac{3}{4}$

$\therefore \quad \cos\theta=\pm\frac{\sqrt{3}}{2}$ but since θ lies in the second quadrant, $\cos\theta$ is negative. $\therefore \quad \cos\theta=-\frac{\sqrt{3}}{2}$

Further $\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{(1/2)}{(-\sqrt{3}/2)}=\frac{-1}{\sqrt{3}}$.

Ex. 2. If $\tan A=2$ and $\pi<A<\frac{3\pi}{2}$, find $\sin A+\cos A$.

Ans. We have $\tan A=2$

$\sec^2 A=1+\tan^2 A=1+2^2=1+4=5$

$\therefore \quad \sec A=\pm\sqrt{5}$ but since A lies in the third quadrant, $\sec A$ is negative. $\therefore \quad \sec A=-\sqrt{5}$.

$\therefore \quad \cos A=\frac{1}{\sec A}=\frac{1}{-\sqrt{5}}=-\frac{1}{\sqrt{5}}$

$\because \quad \tan A=2 \quad \therefore \quad \frac{\sin A}{\cos A}=2 \quad \therefore \quad \sin A=2\cos A$

$\therefore \quad \sin A=2\cos A=2\left(\frac{1}{-\sqrt{5}}\right)=\frac{2}{-\sqrt{5}}$

$\therefore \quad \sin A+\cos A=\frac{-2}{\sqrt{5}}+\left(\frac{-1}{\sqrt{5}}\right)=\frac{-3}{\sqrt{5}}$

Ex. 3. If $6\sec A=3\cot B=2\sqrt{10}$ and $\frac{3\pi}{2}<A<2\pi$, $\pi<B<\frac{3\pi}{2}$, find the value of $\tan A-\operatorname{cosec} B$.

Ans. (i) $6 \sec A = 2\sqrt{10} \quad \therefore \quad \sec A = \frac{\sqrt{10}}{3}$

$$\tan^2 A = \sec^2 A - 1 = \left(\frac{\sqrt{10}}{3}\right)^2 - 1 = \frac{10}{9} - 1 = \frac{1}{9}$$

$\therefore$ $\tan A = \pm \frac{1}{3}$ but since A is in the fourth quadrant, $\tan A = -\frac{1}{3}$.

(ii) $3 \cot B = 2\sqrt{10} \quad \therefore \quad \cot B = \frac{2\sqrt{10}}{3}$

$$\operatorname{cosec}^2 B = 1 + \cot^2 B = 1 + \left(\frac{2\sqrt{10}}{3}\right)^2 = 1 + \frac{40}{9} = \frac{49}{9}$$

$\therefore$ $\operatorname{cosec} B = \pm \frac{7}{3}$ but since B lies in the third quadrant, $\operatorname{cosec} B = -\frac{7}{3}$.

$$\therefore \tan A - \operatorname{cosec} B = -\frac{1}{3} - \left(-\frac{7}{3}\right) = -\frac{1}{3} + \frac{7}{3} = \frac{6}{3} = 2.$$

Exercise 5.7 :

(1) If $\sin \theta = \frac{3}{5}$, $\frac{\pi}{2} < \theta < \pi$ find $\tan \theta$, $\cos \theta$ and $\sec \theta$.

(2) If $\tan A = \frac{1}{\sqrt{7}}$, find the value of $\frac{\operatorname{cosec}^2 A - \sec^2 A}{\operatorname{cosec}^2 A + \sec^2 A}$.

(3) If $\cos A = -1/2$, find $4 \tan^2 A - 3 \operatorname{cosec}^2 A$.

(4) If $\cos \theta = -\frac{3}{5}$ and θ is not in the second quadrant, find $\operatorname{cosec} \theta + \cot \theta$ and $\sec \theta - \tan \theta$.

(5) If $\sin A = \frac{3}{5}$ and A is not in the first quadrant, find $\frac{\sec A - \tan A}{\operatorname{cosec} A - \cot A}$.

(6) If $3 \tan \theta = 1$; $\pi < \theta < \frac{3\pi}{2}$, find $(3 \sec \theta - \operatorname{cosec} \theta)$.

(7) If $\sin A = \frac{3}{5}$ and $\tan B = \frac{1}{2}$ where $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$, find the value of $8 \tan A - \sqrt{5} \sec B$.

(8) If $2 \sin A = 1$ and A is not in the first quadrant and $\sqrt{2} \cos B = 1$, $\frac{3\pi}{2} < B < 2\pi$, find $\frac{\tan A + \tan B}{\cos A + \cos B}$.

(9) If two angles A and B are in the second and third quadrant respectively and $\sin A = -\sin B = -\frac{3}{5}$, find $\sec A - \sec B$.

(10) If $\cos A = \sin B = -\frac{1}{3}$ and $\frac{\pi}{2} < A < \pi$, $\pi < B < \frac{3\pi}{2}$, find $\frac{\tan A - \tan B}{\tan A + \tan B}$.

§ 5·52. Examples :

In the first type of examples on identities, general results concerning any angle θ are required to be proved. In such results, it is assumed that all the ratios used therein exist and extreme cases like $\sin\theta$ or $\cos\theta$ being zero are not taken into consideration.

We use L. H. S. for left hand side and R. H. S. for right hand side.

We also use the following formulae from algebra, whenever required :

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$a^3 - b^3 = (a-b)^3 + 3ab(a-b)$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$a^2 + b^2 = (a+b)^2 - 2ab \text{ or } (a-b)^2 + 2ab.$$

Ex. 1. Prove that $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x$.

Ans. L. H. S. $= \sin^6 x + \cos^6 x$

$= (\sin^2 x)^3 + (\cos^2 x)^3$

$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)$

$= 1^3 - 3\sin^2 x (\cos^2 x)(1)$

$= 1 - 3\sin^2 x \cos^2 x$

$=$ R. H. S.

Ex. 2. Prove : $\cot^2 A \left(\frac{\sec A - 1}{1 + \sin A}\right) + \sec^2 A \left(\frac{\sin A - 1}{1 + \sec A}\right) = 0.$

Ans. L. H. S. $= \frac{\cot^2 A (\sec A - 1)}{1 + \sin A} + \frac{\sec^2 A (\sin A - 1)}{1 + \sec A}$

$= \frac{\cot^2 A (\sec^2 A - 1) + \sec^2 A (\sin^2 A - 1)}{(1 + \sin A)(1 + \sec A)}$

$= \frac{\cot^2 A (\tan^2 A) + \sec^2 A (-\cos^2 A)}{(1 + \sin A)(1 + \sec A)}$

$= \frac{1 - 1}{(1 + \sec A)(1 + \sin A)} = 0 =$ R. H. S.

Ex. 3. Prove that

$$\frac{\tan x}{(1+\tan^2 x)^2}+\frac{\cot x}{(1+\cot^2 x)^2}=\sin x \cos x.$$

Ans. L. H. S. $=\dfrac{\tan x}{(1+\tan^2 x)^2}+\dfrac{\cot x}{(1+\cot^2 x)^2}$

$$=\frac{\tan x}{(\sec^2 x)^2}+\frac{\cot x}{(\operatorname{cosec}^2 x)^2}$$

$$=\frac{\tan x}{\sec^4 x}+\frac{\cot x}{\operatorname{cosec}^4 x}$$

$$=\frac{\left(\dfrac{\sin x}{\cos x}\right)}{\left(\dfrac{1}{\cos^4 x}\right)}+\frac{\left(\dfrac{\cos x}{\sin x}\right)}{\left(\dfrac{1}{\sin^4 x}\right)}$$

$$=\sin x \cos^3 x+\cos x \sin^3 x$$
$$=\sin x \cos x(\cos^2 x+\sin^2 x)$$
$$=\sin x \cos x(1)$$
$$=\text{R. H. S.}$$

Ex. 4. Prove that $\sqrt{\dfrac{1-\cos x}{1+\cos x}}=(\operatorname{cosec} x-\cot x)$

Ans. Here we use the method of multiplying the numerator and denominator of the quantity under the square root by a suitable factor so as to remove the square root. For this note that $1-\cos^2 x=\sin^2 x$. Since the R. H. S. has $\sin x$ as its denominator we proceed as follows :

$$\text{L. H. S.}=\sqrt{\frac{1-\cos x}{1+\cos x}}=\sqrt{\frac{1-\cos x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}}$$

$$=\sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}}=\sqrt{\frac{(1-\cos x)^2}{\sin^2 x}}=\frac{1-\cos x}{\sin x}$$

$$=\frac{1}{\sin x}-\frac{\cos x}{\sin x}=\operatorname{cosec} x-\cot x=\text{R. H. S.}$$

Ex. 5. Prove that

$$\frac{1}{\operatorname{cosec} x-\cot x}-\frac{1}{\sin x}=\frac{1}{\sin x}-\frac{1}{\operatorname{cosec} x+\cot x}$$

Ans. Here it is better to get the terms rearranged. Consequently we write

i. e. to prove that

$$\frac{1}{\operatorname{cosec} x-\cot x}+\frac{1}{\operatorname{cosec} x+\cot x}=\frac{1}{\sin x}+\frac{1}{\sin x}$$

$$\text{Now L. H. S.}=\frac{1}{\operatorname{cosec} x-\cot x}+\frac{1}{\operatorname{cosec} x+\cot x}$$

$$=\frac{(\operatorname{cosec} x+\cot x)+(\operatorname{cosec} x-\cot x)}{\operatorname{cosec}^2 x-\cot^2 x}$$

$$=\frac{2\operatorname{cosec} x}{1}=\frac{2}{\sin x}=\frac{1}{\sin x}+\frac{1}{\sin x}$$

$$\therefore \quad \frac{1}{\operatorname{cosec} x-\cot x}+\frac{1}{\operatorname{cosec} x+\cot x}=\frac{1}{\sin x}+\frac{1}{\sin x}$$

$$\therefore \quad \frac{1}{\operatorname{cosec} x-\cot x}-\frac{1}{\sin x}=\frac{1}{\sin x}-\frac{1}{\operatorname{cosec} x+\cot x}$$

as required.

Exercise 5·8 :

Prove the following :

(1) $(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2=\tan^2\theta+\cot^2\theta+7.$

(2) $\cos^4 A-\sin^4 A=1-2\sin^2 A.$

(3) $\sec^6 A-\tan^6 A=1+3\sec^2 A\tan^2 A.$

(4) $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1.$

(5) $(1+\sin\theta+\cos\theta)^2=2\sin\theta(1+\cos\theta+\cot\theta+\operatorname{cosec}\theta).$

(6) $\frac{\tan x}{1-\cot x}+\frac{\cot x}{1-\tan x}=\sec x\operatorname{cosec} x+1.$

(7) $\sec^4\theta-\sec^2\theta=\tan^2\theta+\tan^4\theta.$

(8) $(1+\cot A+\tan A)(\sin A-\cos A)=\frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A}.$

(9) $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta.$

(10) $\sqrt{\frac{1-\sin x}{1+\sin x}}=\sec x-\tan x.$

(11) $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-1}+\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta+1}=2\sec^2\theta.$

(12) $\frac{\sec x-\tan x}{\sec x+\tan x}=1-2\sec x\tan x+2\tan^2 x.$

(13) $\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}=\sin A+\cos A.$

(14) $\sec^2\theta+\operatorname{cosec}^2\theta=\sec^2\theta\operatorname{cosec}^2\theta.$

(15) $\frac{\cot x\cos x}{\cot x+\cos x}=\frac{\cot x-\cos x}{\cot x\cos x}.$

§ 5·6. Inequalities involving Trigonometric Functions :

(1) $-1\leqslant\sin x\leqslant+1$, $-1\leqslant\cos x\leqslant+1$, for all x.

Since $\sin^2x+\cos^2x=1$ and both $\sin^2x$ and $\cos^2x$ are non-negative, we have $\sin^2x\leqslant1$ and $\cos^2x\leqslant1$.

$\therefore\quad -1\leqslant\sin x\leqslant+1$ and $-1\leqslant\cos x\leqslant+1$.

(2) $\sec^2x\geqslant1$, $\operatorname{cosec}^2x\geqslant1$ for all x

Since $\cos^2x\leqslant1$, $\frac{1}{\cos^2x}\geqslant1$, $\therefore\quad\sec^2x\geqslant1$

Also since $\sin^2x\leqslant1$, $\frac{1}{\sin^2x}\geqslant1$, $\therefore\ \operatorname{cosec}^2x\geqslant1$.

Note: Since $\sec^2x\geqslant1$; $\sec x\geqslant1$ or $\sec x\leqslant-1$.
Also since $\operatorname{cosec}^2x\geqslant1$; $\operatorname{cosec}x\geqslant1$ or $\operatorname{cosec}x\leqslant-1$.

(3) $\sin x<x<\tan x$ for $0<x<\frac{\pi}{2}$:

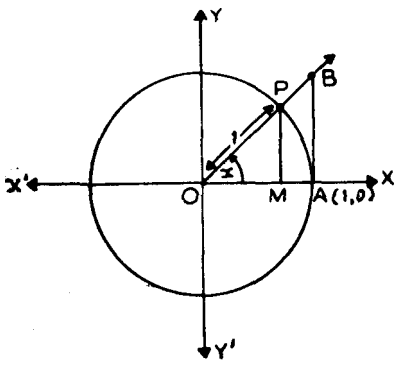


Fig. 5·41

Draw a unit circle with centre O (0, 0) Let A ≡ (1, 0). Let P be the point on the unit circle representing the real number x where $0<x<\frac{\pi}{2}$.

$\therefore\quad P\equiv(\cos x,\ \sin x)$

Also l (arc AP) $=x$ and the circular measure of the angle POA is x radians. Draw seg PM perpendicular seg OA.

$\therefore\ l(PM)=\sin x$ .. (i)

Let the tangent to the circle at A intersect the seg OP (produced) at B. Join seg AP.

In $\triangle$ OAB, $\tan x = \frac{l(AB)}{l(OA)} = \frac{l(AB)}{1}$

$\therefore\ l(AB) = \tan x$ .. (ii)

Area of sector OAP $= \frac{1}{2}r^2x = \frac{1}{2}\cdot 1^2\cdot x = \frac{1}{2}x$

Now Area of $\triangle$ OAP < Area of sector OAP < Area of $\triangle$OAB

$\therefore\ \frac{1}{2}l(OA)\cdot l(PM) < \frac{1}{2}\cdot 1^2\cdot x < \frac{1}{2}l(OA)\cdot l(AB)$

i. e. $\frac{1}{2}\cdot 1\cdot \sin x < \frac{1}{2}\cdot x < \frac{1}{2}\cdot 1\cdot \tan x$ by (i) and (ii).

$\therefore\ \sin x < x < \tan x.$

(4) $\cos x > \sqrt{1-x^2}$ for $0 < x < \frac{\pi}{2}$:

From (3) $\sin x < x$ $\therefore\ \sin^2 x < x^2$ ($\because\ x > 0, \sin x > 0$)

$\therefore\ -\sin^2 x > -x^2,$ $\therefore\ 1-\sin^2 x > 1-x^2$

$\therefore\ \cos^2 x > 1-x^2$ $\therefore\ \cos x > \sqrt{1-x^2}$ ($\because\ \cos x > 0$).

(5) $\cos x < \frac{\sin x}{x} < 1$ for $0 < x < \frac{\pi}{2}$:

From (3) $\sin x < x < \tan x$ $\therefore\ \sin x < x < \frac{\sin x}{\cos x}$

$\therefore\ 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ ($\because\ \sin x > 0$) $\therefore\ 1 > \frac{\sin x}{x} > \cos x$

i. e. $\cos x < \frac{\sin x}{x} < 1.$

(6) $\sqrt{1-x^2} < \frac{\sin x}{x} < 1$ for $0 < x < \frac{\pi}{2}$:

From (4) $\sqrt{1-x^2} < \cos x$ and

from (5) $\cos x < \frac{\sin x}{x} < 1.$

$\therefore\ \sqrt{1-x^2} < \cos x < \frac{\sin x}{x} < 1$

$\therefore\ \sqrt{1-x^2} < \frac{\sin x}{x} < 1.$

(7) $x < \tan x < \frac{x}{\sqrt{1-x^2}}$ for $0 < x < \frac{\pi}{2}$:

From (3) $\sin x < x < \tan x$ $\therefore\ x < \tan x$ ..... (i)

From (4) $\cos x > \sqrt{1-x^2}$ $\therefore\ \frac{1}{\cos x} < \frac{1}{\sqrt{1-x^2}}$ ... (ii)

From (3) $\sin x < x$ and from (ii) $\frac{1}{\cos x} < \frac{1}{\sqrt{1-x^2}}$

$\therefore \sin x \cdot \frac{1}{\cos x} < x \cdot \frac{1}{\sqrt{1-x^2}}$

i.e. $\tan x < x \cdot \frac{1}{\sqrt{1-x^2}}$ .... (iii)

$\therefore$ From (i) and (iii) $x < \tan x < \frac{x}{\sqrt{1-x^2}}$.

§ 5·61. Examples :

Ex. 1. Find $\sin x$ if $2\sin^2 x - 5\sin x - 3 = 0$.

Ans. Since $2\sin^2 x - 5\sin x - 3 = 0$

$\therefore \quad 2\sin^2 x - 6\sin x + \sin x - 3 = 0$

$\therefore 2\sin x(\sin x - 3) + (\sin x - 3) = 0$

$\therefore \quad (\sin x - 3)(2\sin x + 1) = 0$

$\therefore \sin x = 3$ or $\sin x = -\frac{1}{2}$.

But $-1 \leqslant \sin x \leqslant +1$ is always true

$\therefore \sin x = 3$ is not possible $\quad \therefore \sin x = -\frac{1}{2}$.

Ex. 2. Solve $\cos^2 x - 5\cos x + 6 = 0$.

Ans. Since $\cos^2 x - 5\cos x + 6 = 0$

$\therefore \quad \cos^2 x - 2\cos x - 3\cos x + 6 = 0$

$\therefore \cos x(\cos x - 2) - 3(\cos x - 2) = 0$

$\therefore \quad (\cos x - 2)(\cos x - 3) = 0$

$\therefore \cos x = 2$ or $\cos x = 3$

But since $-1 \leqslant \cos x \leqslant +1$, $\cos x$ is not greater than 1.

$\therefore \cos x \neq 2$ and $\cos x \neq 3$

$\therefore$ This equation has no solution.

Exercise 5·9 :

(1) Find $\sin x$ if possible given :
- (a) $\sin^2 x + 4\sin x - 5 = 0$
- (b) $2\sin^2 x + \sin x - 6 = 0$
- (c) $12\sin^2 x + 5\sin x - 3 = 0$
- (d) $2\sin^2 x + \sin x = 0$.

(2) Find $\cos x$ if possible if :
- (a) $3\cos^2 x + 10\cos x + 3 = 0$.
- (b) $2\cos^2 x - 5\cos x + 2 = 0$.
- (c) $\cos^2 x - 12 = 0$.

(3) If k is a real number, is it possible that $\cos x = k + \frac{1}{k}$? Why ?

(4) If p is a real number, is it possible that $\sin x = p^2 - 4p + 6$? Why ?

(5) For what values of k will $\sin x + 1 = k$ be true ?

CHAPTER 6

Compound and Allied Angles

Until now, we have considered trigonometric functions of one angle only. In this Chapter, we shall see how to find trigonometric function of the sum and difference of two angles say A and B, e. g. cos (A − B), sin (A + B), tan (A − B), etc.

§ 6·1. Trigonometric functions of compound and allied angles :

Theorem 6·1 : For all real values of A and B,

$$\cos (A - B) = \cos A \cos B + \sin A \sin B.$$

Proof :

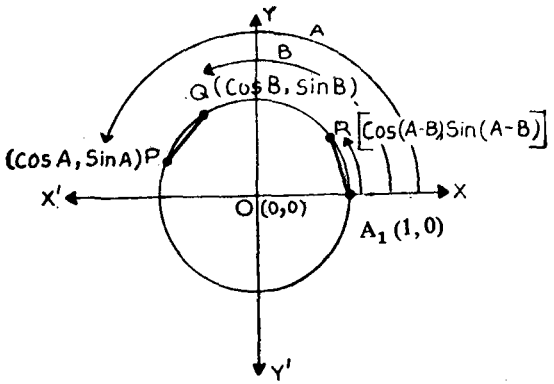


Fig. 6·1

Let $A_1 (1, 0)$ be the point on the unit circle $x^2 + y^2 = 1$ with O (0, 0) as the orlgin.

Let P, Q and R be the points on the unit circle such that the lengths of the arcs A_1P, A_1Q and A_1R are A, B and (A − B) respectively.

$\therefore$ P ≡ (cos A, sin A), Q ≡ (cos B, sin B) and R ≡ (cos (A − B), sin (A − B)).

Now if we consider Q as an initial point and draw an arc on this unit circle of length (A − B), we will get the point P because l [arc A_1Q] + l [arc QP] = B + (A − B) = A.

$\therefore$ length of arc QP = length of arc A_1R.

But we know from Geometry that ' in the same circle if two arcs are congruent, then the chords associated with them are also congruent '.

$\therefore \quad l(A_1R) = l(PQ), \; [l(A_1R)]^2 = [l(PQ)]^2 \quad \ldots\ldots (i)$

Since $A_1 \equiv (1, 0)$, $R \equiv [\cos(A-B), \sin(A-B)]$

$\therefore \quad [l(A_1R)]^2 = [1 - \cos(A-B)]^2 + [0 - \sin(A-B)]^2$

$= 1 - 2\cos(A-B) + \cos^2(A-B) + \sin^2(A-B)$

$= 1 - 2\cos(A-B) + 1$

$= 2 - 2\cos(A-B) \quad \ldots\ldots (ii)$

$\because \quad P \equiv (\cos A, \sin A)$ and $Q \equiv (\cos B, \sin B)$

$[l(PQ)]^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B$

$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$

$= 1 + 1 - 2(\cos A \cos B + \sin A \sin B)$

$= 2 - 2(\cos A \cos B + \sin A \sin B) \quad \ldots\ldots (iii)$

From (i), (ii) and (iii) we get,

$2 - 2\cos(A-B) = 2 - 2(\cos A \cos B + \sin A \sin B)$

$\therefore \quad \cos(A-B) = \cos A \cos B + \sin A \sin B.$

This result is true for all values of A and B.

Note : In the above figure we have shown $0 < B < A < \pi$ only for the sake of simplicity, but as mentioned above, the result is true for all values of A and B.

Theorem 6·2 : For all real numbers A, $\cos(-A) = \cos A$.

Proof: $\cos(A_1 - A_2) = \cos A_1 \cos A_2 + \sin A_1 \sin A_2$. (Theorem 6·1)

Let $A_1 = 0$ and $A_2 = A$

$\therefore \quad \cos(0 - A) = \cos 0 \cdot \cos A + \sin 0 \cdot \sin A$

$\therefore \quad \cos(-A) = 1 \cdot \cos A + 0 \cdot \sin A$

$\therefore \quad \cos(-A) = \cos A.$

Theorem 6·3 : For all real numbers A, $\cos\left(\frac{\pi}{2} - A\right) = \sin A$.

Proof: $\cos(A_1 - A_2) = \cos A_1 \cos A_2 + \sin A_1 \sin A_2$ (Theorem. 6·1)

Let $A_1 = \frac{\pi}{2}$ and $A_2 = A$

$\therefore \cos\left(\frac{\pi}{2} - A\right) = \cos\frac{\pi}{2}\cos A + \sin\frac{\pi}{2} \cdot \sin A$

$= 0 \cdot \cos A + 1 \cdot \sin A$

$= \sin A.$

Theorem 6·4 : For all real numbers A, $\sin\left(\frac{\pi}{2} - A\right) = \cos A$

Proof : $\cos\left(\frac{\pi}{2} - A_1\right) = \sin A_1$ (Theorem 6·3)

Let $A_1 = \frac{\pi}{2} - A.$ $\therefore \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - A\right)\right] = \sin\left(\frac{\pi}{2} - A\right)$

$\therefore \cos A = \sin\left(\frac{\pi}{2} - A\right)$

$\therefore \sin\left(\frac{\pi}{2} - A\right) = \cos A$

Theorem 6·5: For all real numbers A, $\sin\left(\frac{\pi}{2} + A\right) = \cos A.$

Proof : $\sin\left(\frac{\pi}{2} - A_1\right) = \cos(A_1)$ (Theorem 6·4)

Let $A_1 = -A$ $\therefore \sin\left[\frac{\pi}{2} - (-A)\right] = \cos(-A)$

$\therefore \sin\left(\frac{\pi}{2} + A\right) = \cos A$ (Theorem 6·2)

Theorem 6·6 : For all real numbers A, $\sin(-A) = -\sin A.$

Proof: $\sin\left(\frac{\pi}{2} + A_1\right) = \cos A_1$ (Theorm 6·5)

Let $A_1 = -\frac{\pi}{2} - A$

$\therefore \sin\left[\frac{\pi}{2} + \left(-\frac{\pi}{2} - A\right)\right] = \cos\left(-\frac{\pi}{2} - A\right)$

$\therefore \sin(-A) = \cos\left(-\frac{\pi}{2}\right) \cdot \cos A + \sin\left(-\frac{\pi}{2}\right)\sin A$

$= 0 \cdot \cos A + (-1)\sin A$

$= -\sin A.$

Theorem 6·7:

For all real numbers A, $\cos\left(\frac{\pi}{2}+A\right)=-\sin A$.

Proof: $\cos\left(\frac{\pi}{2}+A\right)=\cos\left[\frac{\pi}{2}-(-A)\right]$

$=\cos\frac{\pi}{2}\cdot\cos(-A)+\sin\frac{\pi}{2}\cdot\sin(-A)$

$=0\cdot\cos(-A)+1\cdot(-\sin A)$

$=-\sin A.$

Theorem 6·8: For all real numbers A and B

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

Proof: $\cos(A+B)=\cos[A-(-B)]$

$=\cos A\cdot\cos(-B)+\sin A\sin(-B)$

(Theorem 6·1)

$=\cos A(\cos B)+\sin A(-\sin B)$

(Theorems 6·2 and 6·6)

$=\cos A\cos B-\sin A\sin B.$

Theorem 6·9: For all real numbers A and B

$\sin(A+B)=\sin A\cos B+\cos A\sin B.$

Proof: $\sin(A+B)=\cos\left[\frac{\pi}{2}-(A+B)\right]$ (Theorem 6·3)

$=\cos\left[\left(\frac{\pi}{2}-A\right)-B\right]$

$\therefore\ \sin(A+B)=\cos\left(\frac{\pi}{2}-A\right)\cdot\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B$

(Theorem 6·1)

$=\sin A\cos B+\cos A\sin B$

(Theorems 6·3 and 6·4)

Theorem 6·10: For all real numbers A and B

$\sin(A-B)=\sin A\cos B-\cos A\sin B$

Proof: $\sin(A-B)=\sin[A+(-B)]$

$=\sin A\cos(-B)+\cos A\sin(-B)$

(Theorem. 6·9)

$=\sin A(\cos B)+\cos B(-\sin B)$

(Theorems 6·2 and 6·6)

$=\sin A\cos B-\cos A\sin B.$

Theorem 6·11. For all real numbers A and B

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Proof: $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Theorem 6·12. For all real numbers A and B

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Proof: $\tan(A-B)=\tan[A+(-B)]$

$$=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$$

$$=\frac{\tan A+(-\tan B)}{1-\tan A(-\tan B)}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

We now list together all the theorems proved so far.

(1) $\sin(A+B)=\sin A\cos B+\cos A\sin B.$

(2) $\sin(A-B)=\sin A\cos B-\cos A\sin B.$

(3) $\cos(A+B)=\cos A\cos B-\sin A\sin B.$

(4) $\cos(A-B)=\cos A\cos B+\sin A\sin B.$

(5) $\sin(-A)=-\sin A.$ (6) $\cos(-A)=\cos A.$

(7) $\sin\left(\frac{\pi}{2}-A\right)=\cos A.$ (8) $\cos\left(\frac{\pi}{2}-A\right)=\sin A$

(9) $\sin\left(\frac{\pi}{2}+A\right)=\cos A.$

(10) $\cos\left(\frac{\pi}{2}+A\right)=-\sin A.$

$$(11)\quad \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$(12)\quad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

§ 6.11. Examples :

Ex. 1. Find $\sin 75°$ without using tables.

Ans. $\sin 75° = \sin(45°+30°)$

$$=\sin 45°\cos 30°+\cos 45°\sin 30°$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

In a similar way, we can find the sines and cosines of angles like 15°, 105° etc., which can be expressed as the sum or difference of angles, the values of whose sines and cosines are known. e. g. $15°=45°-30°$, $105°=60°+45°$ etc.

Ex. 2. Prove that

$$\cos\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}+B\right)+\sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}+B\right)=\cos(A-B)$$

$\therefore$ L.H.S. $=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ where $\alpha=\frac{\pi}{4}+A$

and $\beta=\frac{\pi}{4}+B$

$$=\cos(\alpha-\beta)$$

$$=\cos\left[\left(\frac{\pi}{4}+A\right)-\left(\frac{\pi}{4}+B\right)\right]$$

$$=\cos(A-B)$$

$$=\text{R. H. S.}$$

Ex. 3. Prove that

$$\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}=0.$$

Ans. L.H.S. $=\dfrac{(\sin A\cos B-\cos A\sin B)}{\cos A\cos B}$

$$+\frac{(\sin B\cos C-\cos B\sin C)}{\cos B\cos C}+\frac{(\sin C\cos A-\cos C\sin A)}{\cos C\cos A}$$

$$=\frac{\sin A\cos B}{\cos A\cos B}-\frac{\cos A\sin B}{\cos A\cos B}+\frac{\sin B\cos C}{\cos B\cos C}$$

$$-\frac{\cos B\sin C}{\cos B\cos C}+\frac{\sin C\cos A}{\cos C\cos A}-\frac{\cos C\sin A}{\cos C\cos A}$$

$= \tan A - \tan B + \tan B - \tan C + \tan C - \tan A$

$= 0$

$=$ R. H. S.

Ex. 4. Prove that $\dfrac{\tan 5A + \tan 3A}{\tan 5A - \tan 3A} = \dfrac{\sin 8A}{\sin 2A}$

Ans. L.H.S. $= \dfrac{\tan 5A + \tan 3A}{\tan 5A - \tan 3A}$

$$= \frac{\left(\dfrac{\sin 5A}{\cos 5A} + \dfrac{\sin 3A}{\cos 3A}\right)}{\left(\dfrac{\sin 5A}{\cos 5A} - \dfrac{\sin 3A}{\cos 3A}\right)}$$

$$= \frac{\left[\dfrac{\sin 5A \cdot \cos 3A + \cos 5A \cdot \sin 3A}{\cos 5A \cdot \cos 3A}\right]}{\left[\dfrac{\sin 5A \cos 3A - \cos 5A \sin 3A}{\cos 5A \cos 3A}\right]}$$

$$= \frac{\sin 5A \cos 3A + \cos 5A \sin 3A}{\sin 5A \cos 3A - \cos 5A \sin 3A}$$

$$= \frac{\sin (5A + 3A)}{\sin (5A - 3A)} = \frac{\sin 8A}{\sin 2A} = \text{R.H.S.}$$

Ex. 5. Prove that $\sin^2\left(\dfrac{\pi}{4} + \theta\right) + \sin^2\left(\dfrac{\pi}{4} - \theta\right) = 1$

Ans. Now $\left(\dfrac{\pi}{4} + \theta\right) + \left(\dfrac{\pi}{4} - \theta\right) = \dfrac{\pi}{2}$

$$\therefore \frac{\pi}{4} - \theta = \frac{\pi}{2} - \left(\frac{\pi}{4} + \theta\right)$$

$$\therefore \sin\left(\frac{\pi}{4} - \theta\right) = \sin\left\{\frac{\pi}{2} - \left(\frac{\pi}{4} + \theta\right)\right\} = \cos\left(\frac{\pi}{4} + \theta\right)$$

$$\therefore \text{L. H. S.} = \sin^2\left(\frac{\pi}{4} + \theta\right) + \sin^2\left(\frac{\pi}{4} - \theta\right)$$

$$= \sin^2\left(\frac{\pi}{4} + \theta\right) + \cos^2\left(\frac{\pi}{4} + \theta\right) = 1 = \text{R. H. S.}$$

Ex. 6. Prove that $\dfrac{\cos^2 (51°)}{\sin^2 (129°)} - \dfrac{\cos (90° - A)}{\cos (90° + A)} = \sec^2 39°$.

Ans. Now $\cos 51° = \cos (90° - 39°) = \sin 39°$

and $\sin 129° = \sin (90° + 39°) = \cos 39°$.

$$\text{L. H. S.} = \frac{\cos^2 51^\circ}{\sin^2 129^\circ} - \frac{\cos(90^\circ - A)}{\cos(90^\circ + A)}$$

$$= \frac{\sin^2 39^\circ}{\cos^2 39^\circ} - \frac{\sin A}{(-\sin A)}$$

$$= \tan^2 39^\circ + 1 = \sec^2 39^\circ = \text{R. H. S.}$$

Ex. 7. Prove that

$$\cos^2 \frac{\pi}{10} + \cos^2 \frac{2\pi}{5} + \cos^2 \frac{3\pi}{5} + \cos^2 \frac{9\pi}{10} = 2.$$

Ans. $\because \frac{3\pi}{5} = \frac{\pi}{2} + \frac{\pi}{10}$ $\therefore \cos \frac{3\pi}{5} = \cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right) = -\sin \frac{\pi}{10}$

$\because \frac{9\pi}{10} = \frac{\pi}{2} + \frac{2\pi}{5}$ $\therefore \cos\left(\frac{9\pi}{10}\right) = \cos\left(\frac{\pi}{2} + \frac{2\pi}{5}\right) = -\sin \frac{2\pi}{5}$

$$\therefore \text{L. H. S.} = \cos^2 \frac{\pi}{10} + \cos^2 \frac{2\pi}{5} + \left(-\sin \frac{\pi}{10}\right)^2 + \left(-\sin \frac{2\pi}{5}\right)^2$$

$$= \left(\cos^2 \frac{\pi}{10} + \sin^2 \frac{\pi}{10}\right) + \left(\cos^2 \frac{2\pi}{5} + \sin^2 \frac{2\pi}{5}\right)$$

$$= 1 + 1 = 2 = \text{R. H. S.}$$

Ex. 8. Prove that $\sin \frac{5\pi}{6} - 2\sin \frac{2\pi}{3} \cos \frac{5\pi}{6} = 2$

Ans. Now $\frac{5\pi}{6} = \frac{\pi}{2} + \frac{\pi}{3}$ and $\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6}$

$$\therefore \text{L. H. S} = \sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) - 2\sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) \cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$$

$$= \cos \frac{\pi}{3} - 2\cos\left(\frac{\pi}{6}\right) \cdot \left[-\sin \frac{\pi}{3}\right]$$

$$= \frac{1}{2} - 2 \cdot \left(\frac{\sqrt{3}}{2}\right)\left(\frac{-\sqrt{3}}{2}\right)$$

$$= \frac{1}{2} + \frac{3}{2} = 2 = \text{R.H.S.}$$

Exercise 6·1 :

(1) Find (i) sin 15°, (ii) sin 105°, (iii) cos 15°, (iv) cos 105° without using trigonometric tables.

Prove the following :

(2) $\cos\left(\frac{\pi}{4} - A\right)\cos\left(\frac{\pi}{4} - B\right) - \sin\left(\frac{\pi}{4} - A\right)\sin\left(\frac{\pi}{4} - B\right)$

$= \sin(A + B)$

(3) $\sin\left(\frac{\pi}{3}+A\right)\cos\left(\frac{\pi}{6}-B\right)+\cos\left(\frac{\pi}{3}+A\right)\sin\left(\frac{\pi}{6}-B\right)$
$=\cos(A-B)$.

(4) $\cos\alpha\cos(\beta-\alpha)-\sin\alpha\sin(\beta-\alpha)=\cos\beta$.

(5) $\cos\{(n+1)A\}\cos\{(n+2)A\}+\sin\{(n+1)A\}\sin\{(n+2)A\}$
$=\cos A$.

(6) If $2\sin\left(x+\frac{\pi}{3}\right)=\cos\left(x-\frac{\pi}{6}\right)$, then $\tan x+\sqrt{3}=0$.

(7) $\frac{\tan 3\theta+\tan\theta}{\tan 3\theta-\tan\theta}=\frac{\sin 4\theta}{\sin 2\theta}$.

(8) $\frac{\cot 6\theta-\tan 2\theta}{\cot 6\theta+\tan 2\theta}=\frac{\cos 8\theta}{\cos 4\theta}$.

(9) $\frac{\cot\theta\cot 2\theta-1}{\cot\theta\cot 2\theta+1}=\frac{\cos 3\theta}{\cos\theta}$.

(10) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$.

(11) $\frac{\tan(\alpha+\beta)}{\cot(\alpha-\beta)}=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha-\sin^2\beta}$.

(12) $\frac{1}{\tan 3A+\tan A}-\frac{1}{\cot 3A+\cot A}=\cot 4A$.

(13) $\tan\theta\tan 2\theta+1=\sec 2\theta$.

(14) $\tan 2\theta\cot\theta-1=\sec 2\theta$.

(15) $(\cos\theta-\sin\theta)(\cos 2\theta-\sin 2\theta)=\cos\theta-\sin 3\theta$.

(16) $\sqrt{2}\sin\left(\frac{\pi}{4}+A\right)=\sin A+\cos A$.

(17) $2\sin\left(\frac{\pi}{6}+A\right)=\sqrt{3}\sin A+\cos A$.

(18) $\sin 75^\circ-\sin 15^\circ=\cos 105^\circ+\cos 15^\circ$.

(19) $\sin 65^\circ+\cos 125^\circ=\sin(-35^\circ)-\cos(155^\circ)$.

(20) $\cos 55^\circ+\cos 66^\circ+\cos 114^\circ+\cos 125^\circ-\cos 120^\circ=\frac{1}{2}$.

(21) $\tan 10^\circ+\tan 35^\circ+\tan 10^\circ\tan 35^\circ=1$.
[*Hint* : Use $\tan 45^\circ=\tan(35^\circ+10^\circ)$]

(22) $\tan 70^\circ=\tan 20^\circ+2\tan 50^\circ$.

(23) $\tan 24^\circ=\frac{\cos 21^\circ-\sin 21^\circ}{\cos 21^\circ+\sin 21^\circ}$.

(24) $\frac{\sin 20^\circ\cdot\cos^2 70^\circ+\cos 20^\circ\cdot\sin^2 70^\circ}{\sin 20^\circ+\cos 20^\circ}=1-\sin 70^\circ\cos 70^\circ$.

(25) If $A+B=\frac{\pi}{4}$ then $(1+\tan A)(1+\tan B)=2$.

Deduce the value of $\tan\frac{\pi}{8}$.

§ 6·2. Formulae for conversion of sums into products :

Theorem 6·13 :

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2} .$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \cdot \sin \frac{C-D}{2} .$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2} .$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2} .$$

Proof : We know that

$$\sin (A+B) = \sin A \cos B + \cos A \sin B \quad \ldots . \quad (i)$$

and $$\sin (A-B) = \sin A \cos B - \cos A \sin B \quad \ldots . \quad (ii)$$

Adding (i) and (ii) we get

$$\sin (A+B) + \sin (A-B) = 2 \sin A \cos B \quad \ldots . \quad (iii)$$

Subtracting (ii) from (i) we get

$$\sin (A+B) - \sin (A-B) = 2 \cos A \sin B \quad \ldots . \quad iv)$$

Now let $A+B=C$ and $A-B=D$

$\therefore \quad C+D=(A+B)+(A-B)=2A \quad \therefore \quad A = \frac{C+D}{2}$

and $C-D=(A+B)-(A-B)=2B \quad \therefore \quad B=\frac{C-D}{2}$

$\therefore$ (iii) gives : $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2} .$

and (iv) gives : $\sin C - \sin D = 2 \cos \frac{C+D}{2} \cdot \sin \frac{C-D}{2} .$

In a similar way we have

$$\cos (A+B) + \cos (A-B) = 2 \cos A \cos B \quad \ldots . \quad (v)$$

and $$\cos (A+B) - \cos (A-B) = -2 \sin A \sin B \quad \ldots . \quad (vi)$$

Putting $A+B=C$ and $A-B=D$ as before

(v) gives : $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$

and (vi) gives : $\cos C - \cos D = -2 \sin \left(\frac{C+D}{2}\right) \cdot \sin \left(\frac{C-D}{2}\right)$

$$= 2 \sin \left(\frac{C+D}{2}\right) \sin \left[-\frac{C-D}{2}\right]$$

$$= 2 \sin \left(\frac{C+D}{2}\right) \sin \left(\frac{D-C}{2}\right) .$$

as required.

These are called the *factorization formulae* since a sum like sin C + sin D is expressed as a product of sines and cosines.

Theorem 6·14. $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

$2 \cos A \sin B = \sin (A+B) - \sin (A-B)$

$2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

$2 \sin A \sin B = \cos (A-B) - \cos (A+B)$

Proof: $\because \sin (A+B) + \sin (A-B)$

$= (\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B)$

$= 2 \sin A \cos B$

$\therefore \quad 2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

$\because \sin (A+B) - \sin (A-B) = (\sin A \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B)$

$= 2 \cos A \sin B$

$\therefore \quad 2 \cos A \sin B = \sin (A+B) - \sin (A-B)$

$\because \cos (A+B) + \cos (A-B) = (\cos A \cos B - \sin A \sin B) + (\cos A \cos B + \sin A \sin B)$

$= 2 \cos A \cos B$

$\therefore \quad 2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

Similarly $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$.

Using these formulae, a product like $2 \sin A \cos B$ can be written as a sum : $\sin (A+B) + \sin (A-B)$. These may be called as *defactorization formulae.*

§ 6·21. Examples :

Examples on factorization :

Ex. 1. Prove that $\dfrac{\sin 5A + \sin 3A}{\cos 5A + \cos 3A} = \tan 4A$

Ans. L. H. S. $= \dfrac{\sin 5A + \sin 3A}{\cos 5A + \cos 3A}$ Using $\sin C + \sin D$ and $\cos C + \cos D$ formulae :

$$= \frac{2 \sin \left(\frac{5A+3A}{2} \right) \cdot \cos \left(\frac{5A-3A}{2} \right)}{2 \cos \left(\frac{5A+3A}{2} \right) \cdot \cos \left(\frac{5A-3A}{2} \right)}$$

$$= \frac{\sin 4A \cdot \cos A}{\cos 4A \cdot \cos A} = \tan 4A = \text{R. H. S.}$$

Ex. 2. Prove that $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

Ans. Here the terms in L. H. S. must be combined together properly. For this, note that $A + 7A = 3A + 5A$.

$$\text{Hence L. H. S.} = \frac{(\sin 7A + \sin A) + (\sin 5A + \sin 3A)}{(\cos 7A + \cos A) + (\cos 5A + \cos 3A)}$$

$$= \frac{2\sin\left(\frac{7A+A}{2}\right)\cdot\cos\left(\frac{7A-A}{2}\right) + 2\sin\left(\frac{5A+3A}{2}\right)\cdot\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cdot\cos\left(\frac{7A-A}{2}\right) + 2\cos\left(\frac{5A+3A}{2}\right)\cdot\cos\left(\frac{5A-3A}{2}\right)}$$

$$= \frac{2\sin 4A\cos 3A + 2\sin 4A\cos A}{2\cos 4A\cos 3A + 2\cos 4A\cos A}$$

$$= \frac{2\sin 4A\,(\cos 3A + \cos A)}{2\cos 4A\,(\cos 3A + \cos A)}$$

$$= \frac{\sin 4A}{\cos 4A} = \tan 4A = \text{R. H. S.}$$

Ex. 3. Prove that $\sin A + \sin 3A + \sin 5A + \sin 7A = 4\cos A\cdot\cos 2A\cdot\sin 4A$.

Ans. L. H. S. $= (\sin 7A + \sin A) + (\sin 5A + \sin 3A)$ (Apply $\sin C + \sin D$)

$= 2\sin 4A\cos 3A + 2\sin 4A\cos A$

$= 2\sin 4A\,(\cos 3A + \cos A)$

$= 2\sin 4A\,(2\cos 2A\cos A)$

$= 4\cos A\cos 2A\sin 4A = \text{R.H.S.}$

Ex. 4. Prove that $\sin 40° - \cos 70° = \sqrt{3}\cos 80°$.

Ans. Now $\cos 70° = \cos(90° - 20°) = \sin 20°$

$\therefore$ L.H.S. $= \sin 40° - \sin 20°$

$= 2\cos 30°\cdot\sin 10°$

$= 2\cdot\left(\frac{\sqrt{3}}{2}\right)\cdot\sin(90° - 80°)$

$= \sqrt{3}\cos 80° = \text{R.H.S.}$

Examples on defactorization :

Ex. 5. Prove that $\dfrac{\sin 3A\cos 4A - \sin A\cos 2A}{\sin A\sin 4A + \cos A\cos 6A} = \tan 2A$

Ans. Here we rewrite the L.H.S. by multiplying throughout by 2.

$$\text{L.H.S.} = \frac{2\sin 3A\cos 4A - 2\sin A\cos 2A}{2\sin A\sin 4A + 2\cos A\cos 6A}$$

Applying the formulae for 2 sin A cos B etc :

L. H. S.

$$= \frac{[\sin(3A+4A)+\sin(3A-4A)]-[\sin(A+2A)+\sin(A-2A)]}{[\cos(A-4A)-\cos(A+4A)]+[\cos(A+6A)+\cos(A-6A)]}$$

$$= \frac{[\sin 7A+\sin(-A)]-[\sin 3A+\sin(-A)]}{[\cos(-3A)-\cos(5A)]+[\cos 7A+\cos 5A]} = \frac{\sin 7A-\sin 3A}{\cos 7A+\cos 3A}$$

$$= \frac{2\cos 5A\cdot\sin 2A}{2\cos 5A\cdot\cos 2A} = \frac{\sin 2A}{\cos 2A} = \tan 2A = \text{R.H.S.}$$

Ex. 6. Prove that $\cos 38°\cdot\cos 46° - \sin 14°\cdot\sin 22° = \frac{1}{2}\cos 24°$.

Here we use $\cos A \cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$ etc.

$$\text{L.H.S.} = \cos 38°\cos 46° - \sin 14°\sin 22°$$

$$= \frac{\cos(38°+46°)+\cos(38°-46)}{2} - \frac{\cos(14°-22°)-\cos(14°+22°)}{2}$$

$$= \tfrac{1}{2}\{\cos 84° + \cos 8° - \cos 8° + \cos 36°\}$$

$$= \tfrac{1}{2}\{\cos 84° + \cos 36°\}$$

$$= \tfrac{1}{2}\{2\cos 60°\cdot\cos 24°\}$$

$$= \tfrac{1}{2}\cdot 2\cdot\tfrac{1}{2}\cdot\cos 24° = \tfrac{1}{2}\cos 24° = \text{R.H.S.}$$

Exercise 6.2 :

Prove the following :

(1) $\dfrac{\sin 6\theta - \sin 4\theta}{\cos 6\theta + \cos 4\theta} = \tan\theta$

(2) $\dfrac{\sin 7\theta + \sin 3\theta}{\cos 3\theta - \cos 7\theta} = \cot 2\theta$

(3) $\dfrac{\sin A + \sin 2A + \sin 3A + \sin 4A}{\cos A + \cos 2A - \cos 3A - \cos 4A} = \cot A$

(4) $\dfrac{\sin 12A - \sin 4A}{\cos 12A - \cos 4A} = -\cot 8A$

(5) $\dfrac{\sin\theta + \sin 5\theta - \sin 9\theta}{\cos\theta + \cos 5\theta - \cos 9\theta} = \tan 5\theta$

Put $\theta = 12°$ and deduce :

$\sin 12° + \sin 108° = \sqrt{3}(\cos 12° + \cos 108°)$

Is the result valid for $\theta = 18°$? Why?

(6) $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13} = 0.$

(7) $\cos 3\theta - \cos 5\theta + \cos 7\theta - \cos 9\theta = 4 \sin \theta \cos 2\theta \sin 6\theta$.

(8) $\sin 3\theta + \sin 2\theta - \sin \theta = 4 \sin \theta \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$.

(9) $\frac{\sin 2A + \sin 4A - \sin 6A}{\sin 4A + \sin 6A - \sin 2A} = \tan A \tan 3A$

(10) $\sin 10° + \sin 20° + \sin 40° + \sin 50° = 2 \cos 15° \cos 5°$.

(11) If $\sin A + \sin 2A + \sin 3A = \cos A + \cos 2A + \cos 3A$, then either $\tan 2A = 1$ or $\cos A = -\frac{1}{2}$.

(12) $(\cos \alpha + \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \cos^2 \left(\frac{\alpha + \beta}{2}\right)$.

(13) $\frac{\sin 3A \cos 5A - \sin A \cos 7A}{\sin A \sin 7A + \cos 3A \cos 5A} = \tan 2A$

(14) $\sin \frac{3A}{2} \cos \frac{5A}{2} + \cos \frac{7A}{2} \sin \frac{9A}{2} = \sin 6A \cos 2A$.

(15) $\frac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 6A \cot 5A$.

(16) $\sin 18° \cos 39° + \sin 6° \cos 15° = \sin 24° \cos 33°$.

(17) $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200° = -\frac{3}{4}$.

(18) $16 \sin 10° \sin 30° \sin 50° \sin 70° = 1$.

(19) $4 \sin A \sin \left(\frac{\pi}{3} + A\right) \sin \left(\frac{\pi}{3} - A\right) = \sin 3A$

and putting $A = 20°$ deduce: $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$.

§ 6·3. Multiple and submultiple angles :

Theorem 6·15. (i) $\sin 2A = 2 \sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$ .... (*a*)

$= 2 \cos^2 A - 1$ .... (*b*)

$= 1 - 2 \sin^2 A$ .... (*c*)

(iii) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$.

Proof: (i) $\sin 2A = \sin (A + A)$

$= \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$

(ii) $\cos 2A = \cos (A + A)$

$= \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$ .... (*a*)

$= \cos^2 A - (1 - \cos^2 A)$

$= 2 \cos^2 A - 1$ .... (*b*)

$= 2 (1 - \sin^2 A) - 1$

$= 1 - 2 \sin^2 A$ .... (*c*)

(iii) $\tan 2A = \tan(A+A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1 - \tan^2 A}$$

Theorem 6.16: (a) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(b) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$ .... (i)

$= 2\cos^2\frac{A}{2} - 1$ .... (ii)

$= 1 - 2\sin^2\frac{A}{2}$ .... (iii)

(c) $\tan A = \dfrac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$

Proof: $\sin A = \sin\left(\frac{A}{2} + \frac{A}{2}\right) = \sin\frac{A}{2}\cos\frac{A}{2} + \cos\frac{A}{2}\sin\frac{A}{2}$

$= 2\sin\frac{A}{2}\cos\frac{A}{2}$.

$\cos A = \cos\left(\frac{A}{2} + \frac{A}{2}\right) = \cos\frac{A}{2}\cdot\cos\frac{A}{2} - \sin\frac{A}{2}\cdot\sin\frac{A}{2}$

$= \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$ .... (i)

$= \cos^2\frac{A}{2} - \left(1 - \cos^2\frac{A}{2}\right)$

$= 2\cos^2\frac{A}{2} - 1$ .... (ii)

$= 2\left(1 - \sin^2\frac{A}{2}\right) - 1$

$= 1 - 2\sin^2\frac{A}{2}$ .... (iii)

§ 6.31. Examples:

Remember to use $1 + \cos A = 2\cos^2\frac{A}{2}$ and $1 - \cos A = 2\sin^2\frac{A}{2}$ **whenever** they are available.

Ex. 1. Prove that $\frac{\sin A}{1+\cos A} = \tan \frac{A}{2}$

Ans. L. H. S. $= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}}$

$= \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \tan \frac{A}{2} =$ R. H. S.

Ex. 2. Prove that $\operatorname{cosec} 2A + \cot 2A = \cot A$

Ans. L. H. S. $= \frac{1}{\sin 2A} + \frac{\cos 2A}{\sin 2A}$

$= \frac{1+\cos 2A}{\sin 2A} = \frac{2 \cos^2 A}{2 \sin A \cos A}$

$= \frac{\cos A}{\sin A} = \cot A =$ R. H. S.

Ex. 3. Prove that $\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan (A+B)$

Ans. L. H. S. $= \frac{2 \sin^2 A - 2 \sin^2 B}{2 \sin A \cos A - 2 \sin B \cos B}$ (*Note this step*)

$= \frac{(1-\cos 2A) - (1-\cos 2B)}{\sin 2A - \sin 2B}$

$\left(\because 1 - \cos A = 2 \sin^2 \frac{A}{2} \right)$

$= \frac{\cos 2B - \cos 2A}{\sin 2A - \sin 2B}$

$= \frac{2 \sin (A+B) \sin (A-B)}{2 \cos (A+B) \sin (A-B)}$

$= \tan (A+B) =$ R. H. S.

Ex. 4. Prove that $\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A.$

Ans. L. H. S. $= \frac{\sin A + \sin 2A}{(1+\cos 2A) + \cos A}$

$= \frac{\sin A + 2 \sin A \cos A}{2 \cos^2 A + \cos A}$

$= \frac{\sin A (1 + 2 \cos A)}{\cos A (2 \cos A + 1)}$

$= \tan A =$ R. H. S.

Ex. 5. Prove that $\sin 3A = 3 \sin A - 4 \sin^3 A$.

Ans. L. H. S. $= \sin 3A$

$= \sin (2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= (2 \sin A \cos A) \cos A + (1 - 2\sin^2 A) \sin A$

$= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A$

$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$

$= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$

$= 3 \sin A - 4 \sin^3 A =$ R. H. S.

Note: Similarly you can prove : $\cos 3A = 4 \cos^3 A - 3 \cos A$.

Ex. 6. Prove that $\sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A$.

Ans. L. H. S. $= \sin 4A$

$= 2 \sin 2A \cos 2A$

$= 2 (2 \sin A \cos A) (\cos^2 A - \sin^2 A)$

$= 4 \sin A \cos^3 A - 4 \cos A \sin^3 A =$ R. H. S.

Ex. 7. Prove that $\sin \frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}$.

Ans. Let $A = \frac{\pi}{10}$, $\therefore 5A = \frac{\pi}{2}$, $\therefore 2A = \frac{\pi}{2} - 3A$

$\therefore \sin 2A = \sin \left(\frac{\pi}{2} - 3A\right)$, $\therefore \sin 2A = \cos 3A$

$\therefore 2 \sin A \cos A = 4 \cos^3 A - 3 \cos A$

(Refer to note in Ex. 5 above)

$\therefore 2 \sin A \cos A = \cos A [4 \cos^2 A - 3]$

$\therefore 2 \sin A = 4 \cos^2 A - 3 \quad \left(\because A = \frac{\pi}{10}, \cos A \neq 0\right)$

$\therefore 2 \sin A = 4 (1 - \sin^2 A) - 3$

$\therefore 2 \sin A = 4 - 4 \sin^2 A - 3$

$\therefore 4 \sin^2 A + 2 \sin A = 1$

$\therefore 4 \sin^2 A + 2 \sin A + \frac{1}{4} = 1 + \frac{1}{4}$

$\therefore \left(2 \sin A + \frac{1}{2}\right)^2 = \frac{5}{4}$

$\therefore 2 \sin A + \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$

$\therefore 2 \sin A = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$\therefore \sin A = \frac{-1 \pm \sqrt{5}}{4}$

$\therefore$ Either $\sin A = \frac{-1+\sqrt{5}}{4}$ or $\sin A = \frac{-1-\sqrt{5}}{4}$

But $A = \frac{\pi}{10}$ is in the first quadrant

$\therefore \sin A > 0$ $\therefore \sin A = \frac{\sqrt{5}-1}{4}$

Hence $\sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$.

Note : Remember this result as a formulae.

Exercise 6.3 :

Prove the following :

(1) $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot \theta$.

(2) $\tan \theta + \cot \theta = 2 \operatorname{cosec} 2\theta$.

(3) $\tan A - \cot A = -2 \cot 2A$.

(4) $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan \frac{A}{2} \cot \frac{B}{2}$.

(5) $\frac{\sec 8A-1}{\sec 4A-1} = \frac{\tan 8A}{\tan 2A}$.

(6) $\frac{\cos A+\sin A}{\cos A-\sin A} - \frac{\cos A-\sin A}{\cos A+\sin A} = 2 \tan 2A$.

(7) $\cot(A+15°) - \tan(A-15°) = \frac{4 \cos 2A}{1+2 \sin 2A}$.

(8) $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan \frac{\theta}{2}$.

(9) $\frac{\sin(A+3B)+\sin(3A+B)}{\sin 2A+\sin 2B} = 2 \cos(A+B)$.

(10) $(\sec 2A+1)\sqrt{\sec^2 A-1} = \tan 2A$.

(11) $1+\cos^2 2\theta = 2(\cos^4\theta+\sin^4\theta)$.

(12) $\operatorname{cosec} A - 2 \cot 2A \cos A = 2 \sin A$.

(13) $\sec\left(\frac{\pi}{4}+\theta\right) \sec\left(\frac{\pi}{4}-\theta\right) = 2 \sec 2\theta$.

(14) $(1+\tan A-\sec A)(1+\tan A+\sec A) = \sin 2A \sec^2 A$.

(15) $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$.

(16) $\frac{\sin 3B}{\sin B} - \frac{\cos 3B}{\cos B} = 2$.

(17) $\left(\frac{\sin 2A}{\sec A+1}\right)\left(\frac{\sec 2A}{\sec 2A+1}\right) = \tan\left(\frac{A}{2}\right)$.

(18) $\cos 3A = 4\cos^3 A - 3\cos A$.

§ 6·4. Angles of a Triangle :

If A, B, C are angles of a triangle, i.e. $A+B+C=\pi$, then some special results are true for the sines and cosines of these angles. These are proved in the following theorem :

Theorem 6·17: If $A+B+C=\pi$ then

(a) $\sin(A+B) = \sin C$; $\sin(B+C) = \sin A$ and $\sin(C+A) = \sin B$.

(b) $\cos(A+B) = -\cos C$; $\cos(B+C) = -\cos A$ and $\cos(C+A) = -\cos B$.

(c) $\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$; $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ and $\sin\left(\frac{C+A}{2}\right) = \cos\frac{B}{2}$.

(d) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$; $\cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}$ and $\cos\left(\frac{C+A}{2}\right) = \sin\frac{B}{2}$.

Proof: (a) Consider $\sin(A+B)$

$= \sin(\pi - C)$ $(\because A+B+C=\pi)$

$= \sin\pi\cos C - \cos\pi\sin C$ (Theorem 6·10)

$= 0\cdot\cos C - (-1)\cdot\sin C = \sin C$ as required.

Similarly the other formulae in (a) can be proved.

(b) Consider $\cos(A+B)$

$= \cos(\pi - C)$ $(\because A+B+C=\pi)$

$= \cos\pi\cos C + \sin\pi\sin C$ (Theorem 6·1)

$= (-1)\cos C + 0\cdot\sin C = -\cos C$ as required.

Similarly the other formulae in (b) can be proved.

(c) Consider $\sin\left[\frac{A+B}{2}\right]$

$= \sin\left[\frac{\pi - C}{2}\right]$ ($\because A+B+C=\pi$)

$= \sin\left[\frac{\pi}{2} - \frac{C}{2}\right]$

$= \cos\frac{C}{2}$ (Theorem 6·4)

Similarly, the other formulae in (c) can be proved.

(d) Consider $\cos\left[\frac{A+B}{2}\right]$

$= \cos\left[\frac{\pi - C}{C}\right]$ ($\because A+B+C=\pi$).

$= \cos\left[\frac{\pi}{2} - \frac{C}{2}\right]$

$= \sin\frac{C}{2}$ (Theorem 6·3)

Similarly the other formulae in (d) can be proved.

§ 6·1. Examples :

Ex. 1. If $A+B+C=\pi$ prove that

$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

Ans. L. H. S. $= \sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + \sin 2C$

$= 2 \sin (A+B) \cos (A-B) + 2 \sin C \cos C$

$= 2 \sin C \cos (A-B) + 2 \sin C \cos C$

[$\because \sin (A+B) = \sin C$]

$= 2 \sin C [\cos (A-B) + \cos C]$

$= 2 \sin C [\cos (A-B) - \cos (A+B)]$

[$\because \cos (A+B) = -\cos C$]

$= 2 \sin C (2 \sin A \sin B)$ (Theorem 6·14)

$= 4 \sin A \sin B \sin C =$ R. H. S.

Ex. 2. If $A+B+C=\pi$, prove that

$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$

Ans. We use the formula : $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\text{Hence L. H. S.} = \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \sin^2 C$$

$$= \tfrac{1}{2}(2 - \cos 2A - \cos 2B) + \sin^2 C$$
$$= \tfrac{1}{2}[2 - (\cos 2A + \cos 2B)] + \sin^2 C$$
$$= \tfrac{1}{2}[2 - 2\cos(A+B)\cdot\cos(A-B)] + \sin^2 C$$
$$= \tfrac{1}{2}[2 - 2(-\cos C)\cdot\cos(A-B)] + \sin^2 C$$
$$[\because \cos(A+B) = -\cos C]$$
$$= 1 + \cos C\cos(A-B) + \sin^2 C$$
$$= 1 + \cos C\cdot\cos(A-B) + 1 - \cos^2 C$$
$$= 2 + \cos C\cos(A-B) - \cos^2 C$$
$$= 2 + \cos C[\cos(A-B) - \cos C]$$
$$= 2 + \cos C[\cos(A-B) + \cos(A+B)]$$
$$[\because \cos(A+B) = -\cos C]$$
$$= 2 + \cos C(2\cos A\cos B) \text{ (Theorem 6·14)}$$
$$= 2 + 2\cos A\cos B\cos C$$
$$= \text{R. H. S.}$$

Ex. 3. If $A+B+C=\pi$ prove that

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C$$

Ans. Since $A+B=\pi-C$

$$\therefore \quad \tan(A+B) = \tan(\pi - C) = \frac{\tan\pi - \tan C}{1 + \tan\pi\tan C}$$

$$\therefore \quad \tan(A+B) = \frac{0 - \tan C}{1 + 0\cdot\tan C} = -\tan C$$

$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$$

$$\therefore \quad \tan A + \tan B = -\tan C + \tan A\tan B\tan C$$

$$\therefore \quad \tan A + \tan B + \tan C = \tan A\tan B\tan C \text{ as required.}$$

Exercise 6·4 :

If $A+B+C=\pi$, prove the following :

(1) $\cos 2A + \cos 2B + \cos 2C + 1 = -4\cos A\cos B\cos C$.

(2) $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C$.

(3) $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$.

(4) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$.

(5) $\frac{\sin A + \sin B - \sin C}{\sin A - \sin B + \sin C} = \tan\frac{B}{2}\cdot\cot\frac{C}{2}$.

(6) $\cos^2 2A+\cos^2 2B+\cos^2 2C=1+2\cos 2A\cos 2B\cos 2C$.

(7) $\cos^2 C+\cos^2 A-\cos^2 B=1-2\cos B\sin C\sin A$.

(8) $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$.

(9) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$.

(10) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$.

(11) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$.

(12) $\tan\frac{A}{2}.\tan\frac{B}{2}+\tan\frac{B}{2}.\tan\frac{C}{2}+\tan\frac{C}{2}.\tan\frac{A}{2}=1$.

§ 6·5. Periodic functions :

Graphs of some functions can be divided into equal sub-intervals such that the graph of the function in every sub-interva

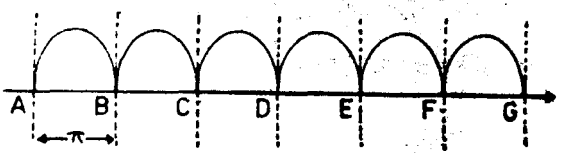


Fig. 6·2

is a repetition of the graph in every other sub-interval. For example, if we refer to the graph in fig. 6·2, we find that the graph in every sub-interval AB, BC, CD, DE, ... etc. is only a repetition of the graph in any of these intervals. Such functions are called *periodic* functions ; the length p of each sub-interval is called a *period* of this function and the smallest positive value of p is called the *fundamental period* of this function, e. g. the length of every sub-interval of the graph in Fig. 6·2 is π, i. e. $AB=BC=CD=\ldots.=\pi$, then π is called a period of this function. Further the graph in the sub interval AC (of the length 2π) is also repeated in every similar sub-interval such as CE, EG, etc. Hence 2π is also a period of this function. Similarly, 3π, 4π, 5π .... in general $n\pi$, ($n\in$ I, $n\neq 0$) are all periods of this function but the smallest positive value of p in this case is π and hence π is the *fundamental period* of this function. Similarly, for the graph of the function given in fig. 6·3 (shown on page 115), 2π, 4π, 6π, ... $2n\pi$, ($n\in$ I, $n\neq 0$) are all periods of this function but the smallest positive value of p, i. e. 2π is the fundamental period.

Hence if p is the fundamental period of any function f, then p,

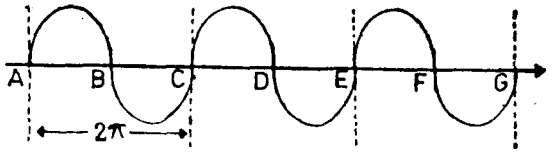


Fig. 6·3

$2p, 3p, 4p, \ldots . -p, -2p, -3p, \ldots .$, i. e. in general np where $n \neq 0$ and $n \in$ I are also periods of this function.

Definition: If f is a function whose domain is D, where D $\subseteq$ R such that for some $p \in$ R, $p \neq 0$, the values of $(x+p)$ and $(x-p)$ are in D for each $x \in$ D and $f(x+p)=f(x)$ for all $x \in$ D then f is *periodic* with *period p*. The smallest positive value of p for which the function f is periodic, is called the *fundamental period* of f.

As observed earlier, if a function f is periodic with period p then it is also periodic with periods $2p, 3p, 4p, \ldots .$, $-p$, $-2p$, $-3p \ldots .$, i. e. in general np where $n \in$ I and $n \neq 0$.

Hence, if p is the fundamental period of a function $f(x)$ then $f(x+np)=f(x)$ where $n \in$ I. Here we do not say that n should not be equal to zero because $f(x+np)=f(x)$ is true when $n=0$. $\therefore$ $f(x+np)=f(x)$ for $n \in$ I.

§ 6·51. Periodicity of trigonometric functions :

Consider the function $f(x)=\sin x$, $x \in$ R.

Then $\sin(2\pi+x)=\sin 2\pi \cos x+\cos 2\pi \sin x$
$=0\cdot\cos x+1\cdot\sin x=\sin x$.

Hence $\sin x$ is periodic and 2π is a period. We can prove similarly that $4\pi, 6\pi, 8\pi, \ldots .$ are also periods.

Since $\sin(\pi+x)=-\sin x$, π is not a period of $\sin x$, $\therefore$ 2π is the smallest positive period.

Hence $f(x)=\sin x$ is periodic with *fundamental period* 2π.
.... (i)

Similarly, we can show that $f(x)=\cos x$ is also periodic with *fundamental period* 2π .... (ii)

Now consider $f(x)=\tan x$. $x \in$ R, $x \neq (2n+1)\frac{\pi}{2}$, $n \in$ I.

Then $\tan(2\pi+x)=\dfrac{\sin(2\pi+x)}{\cos(2\pi+x)}=\dfrac{\sin x}{\cos x}=\tan x$.

$\therefore$ $\tan x$ is periodic with 2π as a period. *Similarly*, 4π, 6π, 8π, ... i. e. in general, $2n\pi$, $n \in N$ are periods of $\tan x$.

But consider $\tan(\pi+x) = \dfrac{\sin(\pi+x)}{\cos(\pi+x)} = \dfrac{-\sin x}{-\cos x} = \tan x$.

Hence π is also a period of $\tan x$. Now let us check whether $\dfrac{\pi}{2}$ is a period of $\tan x$ or not.

Since $\tan\left(\dfrac{\pi}{2}+x\right) = \dfrac{\sin\left(\dfrac{\pi}{2}+x\right)}{\cos\left(\dfrac{\pi}{2}+x\right)} = \dfrac{\cos x}{\sin x} = \cot x$, $\dfrac{\pi}{2}$ is not a period of $\tan x$. Hence π is the smallest positive period. Hence $\tan x$ is periodic with fundamental period π .... (iii)

Theorem 6·18. If $f(x) = \sin kx$ where $k > 0$ then $f(x)$ is periodic with fundamental period $\dfrac{2\pi}{k}$.

Proof: We know that $\sin(\theta) = \sin(\theta + 2\pi)$.

Hence $f(x) = \sin(kx) = \sin(kx + 2\pi)$ .... (i)

$$= \sin\left[k \cdot \left(x + \frac{2\pi}{k}\right)\right]$$

$$= f\left(x + \frac{2\pi}{k}\right)$$

Hence $f(x)$ is periodic with $\dfrac{2\pi}{k}$ as the fundamental period since we have used the fundamental period 2π of the sine function in (i). Here $f(x) = \sin kx$ is periodic with fundamental period $\dfrac{2\pi}{k}$ as required.

Similarly, we can prove that $f(x) = \cos kx$, $k > 0$, is periodic with fundamental period $\dfrac{2\pi}{k}$ and $f(x) = \tan kx$, $k > 0$, is periodic with fundamental period $\dfrac{\pi}{k}$.

These results are very important in solving examples.

§ 6·52. Examples:

Ex. 1. Find whether $f(x) = \sin(3x+2)$ is periodic. If periodic, find its fundamental period ?

Ans. Since $\sin\theta = \sin(2\pi+\theta)$, we have

$$f(x) = \sin(3x+2) = \sin[(3x+2)+2\pi]$$
$$= \sin\left[(3x+2\pi)+2\right] = \sin\left[3\left(x+\frac{2\pi}{3}\right)+2\right]$$
$$= f\left(x+\frac{2\pi}{3}\right).$$

$\therefore$ $f(x)$ is periodic with period $\frac{2\pi}{3}$. Further $\frac{2\pi}{3}$ is the fundamental period since we have used the fundamental period 2π for the sine function.

Ex. 2. Find a function whose fundamental period is 4.

Ans. Let $f(x) = \sin kx$ be the required function

$\therefore$ fundamental period $= \frac{2\pi}{k}$ (Theorem 6·18)

Since we require the fundamental period to be 4, $\frac{2\pi}{k} = 4$.

$$\therefore\ k = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Hence $\sin\left(\frac{\pi}{2}x\right)$ is the required function which has fundamental period 4.

Ex. 3. Is the function $f(x) = \sin^2 x$ periodic? If so, what is the fundamental period?

Ans. Since $f(x) = \sin^2 x = (\sin x)^2 = [\sin(2\pi+x)]^2$
$= \sin^2(2\pi+x) = f(2\pi+x)$

$\therefore$ $f(x)$ is periodic with a period 2π. Obviously, 4π, 6π, 8π, ... etc. are also periods of $f(x)$. Now let us check whether π is a period or not.

$$f(x+\pi) = \sin^2(x+\pi) = [\sin(x+\pi)]^2$$
$$= (-\sin x)^2 = \sin^2 x = f(x).$$

$\therefore$ π is also a period. Hence let us check wheter $\frac{\pi}{2}$ is a period or not.

$$f\left(x+\frac{\pi}{2}\right) = \sin^2\left(x+\frac{\pi}{2}\right) = \cos^2 x.$$

$\therefore$ $\frac{\pi}{2}$ is not a period. $\therefore$ π is the smallest positive period.

$\therefore$ $f(x)$ is periodic with fundamental period π.

Ex. 4. Is the function $f(x) = \sin(x^2)$ periodic ?

Ans. $f(x) = \sin(x^2) = \sin(x^2 + 2\pi)$.

Now, if f is periodic with a period, say p, then we must have $\sin(x^2 + 2\pi) = f(x + p)$.

But $f(x + p) = \sin[(x + p)^2]$.

Hence we must find p such that $(x + p)^2 = x^2 + 2\pi$.

$\therefore \quad x^2 + 2px + p^2 = x^2 + 2\pi$

$\therefore \quad 2px + p^2 = 2\pi$ which does not give a constant value of p.

Hence the function $f(x) = \sin(x^2)$ is not periodic.

Exercise 6·5 :

(1) Find out whether the following functions are periodic and in case they are, find their fundamental periods.

(a) $f(x) = \sin(4x + 5)$; (b) $f(x) = \cos(6 - 7x)$;
(c) $f(x) = \tan(3x + 1)$; (d) $f(x) = \cos^2 x$;
(e) $f(x) = \tan^2 x$; (f) $f(x) = \sqrt{\sin x}$;
(g) $f(x) = \cos(x^2)$; (h) $f(x) = \tan(x^2)$.

(2) Find a function with fundamental period
(a) 2, (b) π, (c) 4π, (d) 7, (e) π^2, (f) $(\pi + 1)$.

§ 6·6. Graphs of trigonometric functions :

We shall now draw the graphs of the three trigonometric functions $\sin x$, $\cos x$ and $\tan x$. For this we shall assume the following two approximations : $\sqrt{2} = 1{\cdot}42$ and $\sqrt{3} = 1{\cdot}74$, e. g. $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1{\cdot}42}{2} = 0{\cdot}71$.

§ 6·61. Graph of sin x :

We shall draw the graph of $y = \sin x$ where $-\pi \leqslant x \leqslant \pi$

We tabulate the values of x and y as follows :

x	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-3\pi}{4}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$
y	0	−0·5	−0·71	−0·87	−1·0	−0·87	−0·71	−0·5

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y	0	0·5	0·71	0·87	1·0	0·87	0·71	0·5	0

When these points are plotted on a suitable scale, we get the following graph :

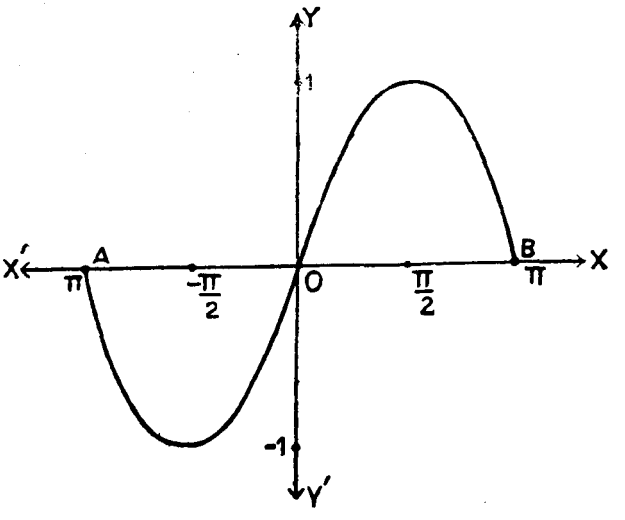


Fig. 6·4

Extension of the graph of sin x : Since $\sin x$ is periodic with period 2π, this graph from A to B, which covers an interval of length 2π, will be repeated in the intervals π to 3π, 3π to 5π etc. and also in -3π to $-\pi$, -5π to -3π etc. The general graph of $y = \sin x$ is as follows :

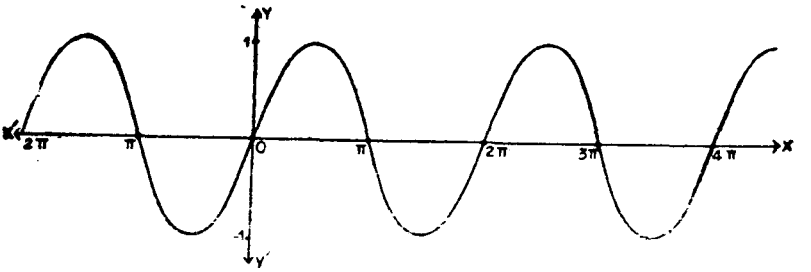


Fig. 6·5

§ 6·62. **Graph of cos x :** We shall draw the graph of $y = \cos x$ where $-\pi \leqslant x \leqslant \pi$

We tabulate the values of x and y as follows :

x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
y	-1·0	-0·87	-0·71	-0·5	0	0·5	0·71	0·87	1·0

x	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y	0·87	0·71	0·5	0	-0·5	-0·71	-0·87	-1·0

When these points are plotted on a suitable scale, we get the graph shown in fig. 6·6

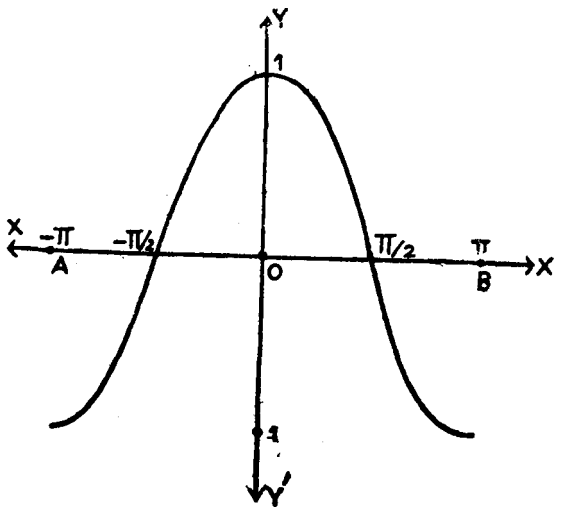


Fig. 6·6

Extension of the graph of cos x : Since cos x is periodic with period 2π, this graph from A to B, which covers an interval of length 2π, will be repeated in the intervals π to 3π, 3π to 5π etc. and also in -3π to $-\pi$, -5π to -3π etc. The general graph of $y = \cos x$ is as follows :

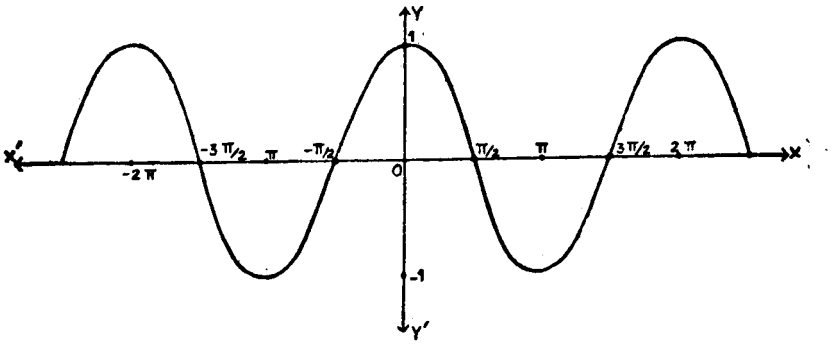


Fig. 6·7

§ 6·63. Graph of tan x :

We shall draw the graph of $f(x) = \tan x$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

For this, we note that tan x does not exist at $x = \pm \frac{\pi}{2}$ but as x increases from 0 to $\frac{\pi}{2}$, since sin x increases from 0 to 1 and cos x decreases from 1 to 0, $\tan x = \frac{\sin x}{\cos x}$ will increase indefinitely as x approaches $\frac{\pi}{2}$. Similarly as x decreases from 0 to $-\frac{\pi}{2}$, tan x decreases indefinitely.

We tabulate the values of x and y as follows :

x	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	−1·73	−1·0	−0·58	0	0·58	1·0	1·73

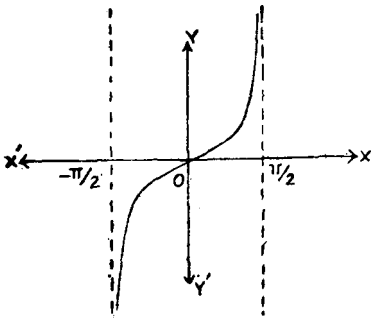


Fig. 6·8

When these points are plotted on a suitable scale we get the graph shown in fig. 6·8.

Extension of the graph of tan x : Since tan x is periodic with period π, t his graph in the interval $-\frac{\pi}{2}<x<\frac{\pi}{2}$, which is an interval of length π, will be repeated in $\frac{\pi}{2}<x<\frac{3\pi}{2}$, $\frac{3\pi}{2}<x<\frac{5\pi}{2}$ etc. and also in $\frac{-3\pi}{2}<x<-\frac{\pi}{2}$; $\frac{-5\pi}{2}<x<\frac{-3\pi}{2}$ etc. The general graph of $y=\tan x$ is as follows :

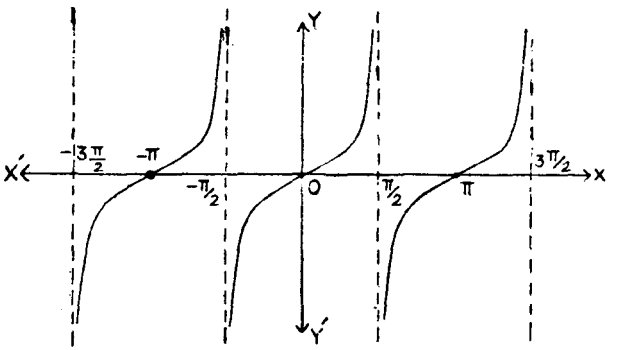


Fig. 6·9

Exercise 6·6 :

(1) Draw the graph of $y=\sin x$ $\quad 0 \leqslant x \leqslant 2\pi$

(2) Draw the graph of $y=\cos x$ $\quad 0 \leqslant x \leqslant 2\pi$

(3) Draw the graph of $y=\tan x$ $\quad \frac{\pi}{2} < x \leqslant \pi$

CHAPTER 7

The Triangle

§ 7·1. Right Triangles:

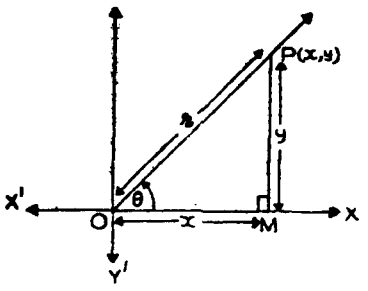


Fig. 7·1

Let $P(x, y)$ be any point other than origin $0(0, 0)$ on the terminal side of an angle θ, where θ is in first quadrant and in standard position. Draw PM perpendicular to the x axis. Then $PM = y$, $OM = x$, and $OP = +\sqrt{x^2+y^2} = r$ where $r > 0$.

Since θ is in the first quadrant, we have a right triangle OPM, where the side opposite to the angle θ is y, the side adjacent to the angle θ is x and the hypotenuse is r ($r > 0$). Thus in relation to a right triangle, we see that the ratios of the sides of this right triangle such as $\frac{y}{r}$, $\frac{x}{r}$, $\frac{y}{x}$ etc. are the trigonometrical ratios $\sin\theta$, $\cos\theta$, $\tan\theta$ etc. Thus, for a right triangle, we can define trigonometrical ratios as follows:

$$\sin\theta = \frac{\text{length of side opposite to } \theta}{\text{length of hypotenuse}}$$

$$\cos\theta = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$$

$$\tan\theta = \frac{\text{length of side opposite to } \theta}{\text{length of side adjacent to } \theta}$$

$$\operatorname{cosec}\theta = \frac{\text{length of hypotenuse}}{\text{length of side opposite to } \theta}$$

$$\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to } \theta}$$

$$\cot\theta = \frac{\text{length of side adjacent to } \theta}{\text{length of side opposite to } \theta}$$

One advantage of defining trigonometrical ratios in terms of the sides of a right triangle is that the angle θ need not be in standard position and can be applied to the sides of any right triangle such as those given in fig. 7·2

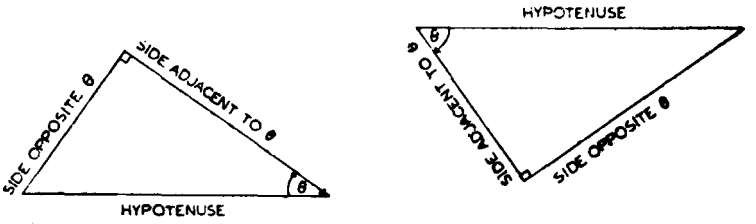


Fig. 7·2

If measures of some angle of a right triangle and the length of some side is given or if lengths of any two sides are given, then with the help of the above definitions of trigonometric ratios, we can find the length of the other sides and the measure of the other angle. This is called " solving a right triangle " and the result so obtained is called the " solution of the right triangle ". From Geometry, we already know that a right triangle is uniquely determined when we have (i) a right angle, and (ii) either any two sides or one side and one angle. Hence we can always " solve a right triangle ", i. e., we can find the length of the remaining side and measure of the remaining angle if besides a right angle (i) the lengths of two sides are given or (ii) length of one side and measure of any of the other angles is given.

§ 7·11. Examples :

Ex. 1. In $\triangle$ ABC, $m\angle B = 90°$, l (BC) = 1, l (AC) = 2, find $m\angle A$, $m\angle C$ and l (AB).

Ans.

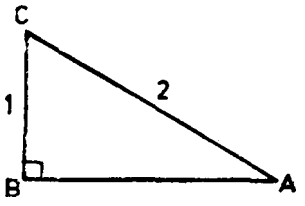


Fig. 7·3

$$\cos C = \frac{l(BC)}{l(AC)} = \frac{1}{2} = \cos 60°$$

$$\therefore \cos C = \cos 60°$$

$$m\angle C = 60° \quad \ldots\ (i)$$

$$m\angle A = 30° \quad \ldots\ (ii)$$

$$l(AB) = \sqrt{2^2 - 1} = \sqrt{3} \quad \ldots\ (iii)$$

Ex. 2. In $\triangle ABC$, $m\angle A = 90°$, $m\angle C = \frac{\pi}{6}$, $l(AC) = \sqrt{3}$, find $l(AB)$ and $l(BC)$.

Ans.

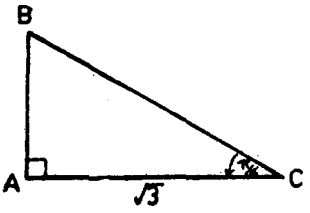


Fig. 7·4

$$\tan\frac{\pi}{6} = \frac{l(AB)}{\sqrt{3}}$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{l(AB)}{\sqrt{3}}$$

$$\therefore l(AB) = 1$$

$$\therefore l(BC) = \sqrt{1^2 + (\sqrt{3})^2}$$

$$= \sqrt{4} = 2.$$

Exercise 7·1 :

(1) In $\triangle ABC$, $m\angle A = 90°$ and $l(AB) = 2$, $l(AC) = 2$, find $m\angle B$ and $l(BC)$.

(2) In $\triangle ABC$, $m\angle A = 90°$, $l(BC) = 4$, $l(AB) = 2$, find $m\angle B$ and $l(AC)$.

(3) Triangle ABC is an equilateral triangle with side of length 6 cms. If seg AD is the median of the triangle meeting seg BC at D, find $l(AD)$.

(4) Triangle ABC is isosceles with $m\angle B = m\angle C = \frac{\pi}{6}$, AD is the median meeting BC at D. Find $l(AD)$ and $l(AB)$ given that $l(BC) = 6$ cms.

§ 7·2. Any Triangle :

The method mentioned above in § 7·1 of "solving the right triangle" is obviously applicable to any right triangle only but not to any other triangle. To 'solve any other triangle' we have to prove some rules known as Sine Rule, Cosine Rule etc. which give relations between the sides and angles of *any* triangle and with the help of which we can solve the triangles which may not be right triangles.

Before proving these rules, we will use some standard notations to represent the angles and the sides of a $\triangle$ ABC. We shall denote the measures of the angles $\angle CAB$, $\angle ABC$ and $\angle BCA$ as A, B and C respectively and the lengths of the sides opposite to the angles A, B and C i.e. lengths of segs BC, CA and AB as a, b and c respectively as in fig. 7·5.

All the elements are to be taken as positive, i. e. the angles or the sides cannot be negative or zero. Further since A, B, C are

the measures of the angles of $\triangle$ ABC, we have $A+B+C=180°$, or $A+B+C=\pi$.

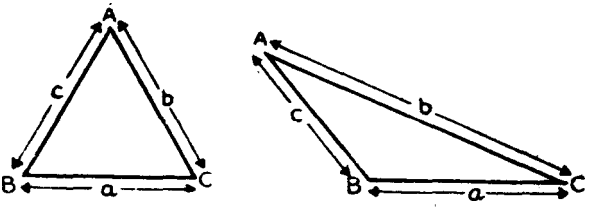


Fig. 7·5

§ 7·21. Cosine Rule :

If A, B and C are the measures of the angles of a $\triangle$ ABC and a, b and c are the lengths of the sides opposite to the angles A, B and C respectively, then

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \quad \text{or} \quad a^2 = b^2+c^2-2bc\cos A$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} \quad \text{or} \quad b^2 = c^2+a^2-2ca\cos B$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \quad \text{or} \quad c^2 = a^2+b^2-2ab\cos C.$$

Proof :

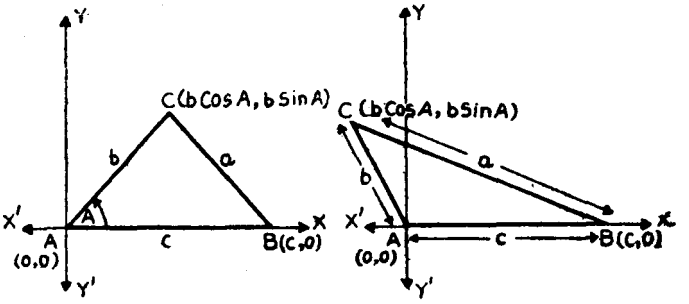


Fig. 7·6

Let us place the angle A of $\triangle$ ABC is standard position, i. e. A as the origin and AB along positive x axis. Further $m\angle A < 180°$ $\therefore$ the point C is above the x axis.

Also $l(AB) = c$, $\therefore$ $B \equiv (c, 0)$.

Let $C \equiv (x, y)$. Since $l(AC) = b$ we have $\cos A = \frac{x}{b}$ and

$\sin A = \frac{y}{b}$. $\therefore\ x = b\cos A$ and $y = b\sin A$.

$\therefore\ C \equiv (b\cos A,\ b\sin A)$,

Now $l(BC) = a$ $\quad\therefore\ a^2 = [l(BC)]^2$

$\therefore\ a^2 = (c - b\cos A)^2 + (b\sin A)^2$

$= c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A$

$= c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A)$

$= c^2 - 2bc\cos A + b^2 \cdot (1)$

$= b^2 + c^2 - 2bc\cos A$ .... (i)

$\therefore\ 2bc\cos A = b^2 + c^2 - a^2$

$\therefore\ \cos A = \frac{b^2 + c^2 - a^2}{2bc}$ .... (ii) ($\because\ b \neq 0, c \neq 0$)

Similarly we can prove that,

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}, \text{ i. e., } b^2 = c^2 + a^2 - 2ca\cos B$$

and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, i. e., $c^2 = a^2 + b^2 - 2ab\cos C$

Corollaries :

(1) If $b^2 + c^2 > a^2$, i. e. $\cos A > 0$, then $\angle A$ is acute.

(2) If $b^2 + c^2 < a^2$, i. e. $\cos A < 0$, then $\angle A$ is obtuse.

(3) If $b^2 + c^2 = a^2$, i. e. $\cos A = 0$, then $\angle A$ is right angle.

(4) If $A = \frac{\pi}{2}$, then $\cos A = 0$, $\therefore\ \frac{b^2 + c^2 - a^2}{2bc} = 0$

$\therefore\ b^2 + c^2 - a^2 = 0$ $\quad\therefore\ b^2 + c^2 = a^2$.

(5) $a = b\cos C + c\cos B$

$b = c\cos A + a\cos C$

$c = a\cos B + b\cos A$.

This rule is known as "Projection Rule".

Proof : $b\cos C + c\cos B$

$$= b \cdot \frac{a^2 + b^2 - c^2}{2ab} + c \cdot \frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{(a^2 + b^2 - c^2) + (c^2 + a^2 - b^2)}{2a}$$

$$= \frac{2a^2}{2a} = a.$$

$\therefore\ a = b\cos C + c\cos B$.

Similarly $b = c \cos A + a \cos C$

and $c = a \cos B + b \cos A$

Applications :

(1) If the lengths of all the sides of a triangle are given, the measures of all the angles can be found.

(2) If the lengths of any two sides (e. g., b and c) and the measure of the included angle (A) is given then the length of the remaining side a can be found.

§ 7·22. Sine Rule :

If A, B and C are the measures of the angles of a triangle and a, b and c are the lengths of the sides opposite to the angles A, B and C respectively, then $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Proof :

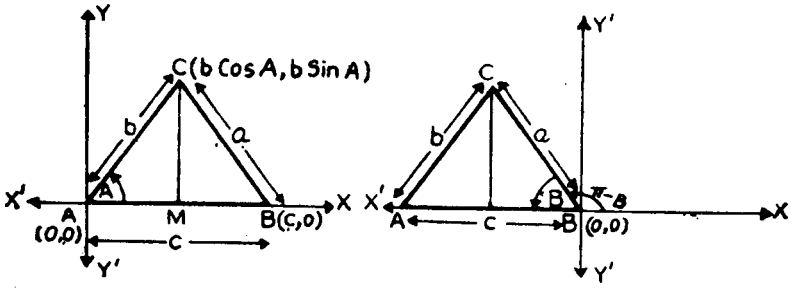


Fig. 7·7

Place the angle A of $\triangle ABC$ in standard position, with A at the origin (0, 0) and AB along the positive x axis. Since angle $A < 180°$, C is above the x axis. Draw CM $\perp$ x axis meeting it at M.

$l(AB) = c$ $\quad \therefore$ B $\equiv (c, 0)$

$l(AC) = b$ $\quad \therefore$ C $\equiv (b \cos A, b \sin A)$

$\therefore l(CM) = b \sin A$ .... (i)

Now select AB along the x axis such that A is on the negative x axis and B at the origin. $\therefore$ B is (0, 0), and CB makes an angle of $(\pi - B)$ with the positive x axis.

$\therefore$ C $\equiv (a \cos(\pi - B), a \sin(\pi - B))$

$\equiv (-a \cos B, a \sin B)$

$\therefore l(CM) = a \sin B$ .... (ii)

From (i) and (ii) we get

$$b \sin A = a \sin B \qquad \therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly we can prove that $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} .$$

§ 7·23. Examples :

Ex. 1. Assuming the cosine Rule, prove the Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} .$$

Ans. Given $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\therefore \quad \sin^2 A = 1 - \cos^2 A$$

$$= 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2$$

$$= 1 - \frac{b^4 + c^4 + a^4 + 2b\ c^2 - 2a^2\ c^2 - 2a^2\ b^2}{4b^2\ c^2}$$

$$= \frac{4b^2\ c^2 - (b^4 + c^4 + a^4 + 2b^2\ c^2 - 2a^2\ c^2 - 2a^2\ b^2)}{4a^2\ c^2}$$

$$= \frac{2a^2\ b^2 + 2b^2\ c^2 + 2c^2\ a^2 - (a^4 + b^4 + c^4)}{4b^2\ c^2}$$

$$\therefore \quad \frac{\sin^2 A}{a^2} = \frac{2a^2\ b^2 + 2b^2\ c^2 + 2c^2\ a^2 - (a^4 + b^4 + c^4)}{4a^2\ b^2\ c^2}$$

Similarly $\frac{\sin^2 B}{b^2} = \frac{2a^2\ b^2 + 2b^2\ c^2 + 2c^2\ a^2 - (a^4 + b^4 + c^4)}{4a^2\ b^2\ c^2}$

and $\frac{\sin^2 C}{c^2} = \frac{2a^2\ b^2 + 2b^2\ c^2 + 2c^2\ a^2 - (a^4 + b^4 + c^4)}{4a^2\ b^2\ c^2}$

$$\therefore \quad \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2} \qquad \ldots\ldots \qquad \ldots\ldots \quad (i)$$

Since each angle A, B, C is less than 180°, sin A, sin B and sin C are always positive. Hence, taking only positive square root from (i) we get $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$$\text{i. e.} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} .$$

Ex. 2. Show with usual notations that the area of a $\triangle$ ABC is $\frac{1}{2}\, bc \sin A$.

Ans.

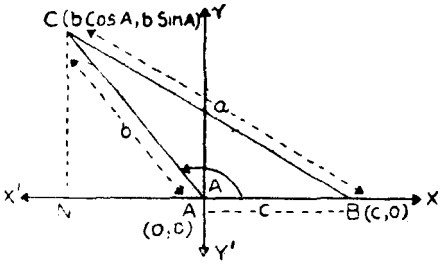


Fig. 7·8

Place the angle A in standard position taking A at the origin and AB along positive x axis. Draw CN $\perp$ x axis.

$\because$ $l(AB) = c$

$\therefore$ B $\equiv (c, 0)$

$\because$ $l(AC) = b$ $\quad\therefore\quad$ C $\equiv (b \cos A,\ b \sin A)$

$\therefore$ $l(CN) = b \sin A$.

$\therefore$ Area of $\triangle$ ABC $= \frac{1}{2} \cdot l(AB) \cdot l(CN)$

$= \frac{1}{2} \cdot c \cdot b \sin A$

$= \frac{1}{2}\, bc \sin A$ .... **Ans.**

Note that we can also prove that

Area of $\triangle$ ABC $= \frac{1}{2}\, ca \sin B = \frac{1}{2}\, ab \sin C$.

Ex. 3. In Δ ABC prove that :

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right) \cot\frac{A}{2}$$

Ans. This result is known as Napier's Analogy. In such an example note that L. H. S. contains only angles. Hence the sides on the R. H. S. should be converted into angles as follows :

Let $\quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

$\therefore$ $a = k \sin A$, $\quad b = k \sin B$ and $\quad c = k \sin C$

$\therefore$ R. H. S. $= \left(\frac{k \sin B - k \sin C}{k \sin B + k \sin C}\right) \cdot \cot\frac{A}{2}$

$= \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)} \cdot \cot\frac{A}{2}$

$= \frac{2 \cos\left(\frac{B+C}{2}\right) \sin\left(\frac{B-C}{2}\right)}{2 \sin\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right)} \cdot \cot\frac{A}{2}$

$$= \cot\left(\frac{B+C}{2}\right)\cdot\cot\frac{A}{2}\cdot\tan\left(\frac{B-C}{2}\right)$$

$$= \cot\left(\frac{\pi}{2}-\frac{A}{2}\right)\cdot\cot\frac{A}{2}\cdot\tan\left(\frac{B-C}{2}\right)$$

$$\left[\because \quad \frac{B+C}{2} = \frac{\pi-A}{2} = \frac{\pi}{2}-\frac{A}{2}\right]$$

$$= \tan\frac{A}{2}\cdot\cot\frac{A}{2}\cdot\tan\left(\frac{B-C}{2}\right)$$

$$= \tan\left(\frac{B-C}{2}\right) = \text{L. H. S.}$$

$$\therefore \quad \tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cdot\cot\frac{A}{2}.$$

Ex. 4. In $\triangle$ ABC, prove :

$$\cos\left(\frac{B-C}{2}\right) = \left(\frac{b+c}{a}\right)\cdot\sin\left(\frac{A}{2}\right)$$

Ans. Now $\quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin B} = k$ say

$$\therefore \quad \text{R. H. S.} = \frac{k\sin B + k\sin C}{k\sin A}\cdot\sin\frac{A}{2}$$

$$= \frac{k(\sin B+\sin C)}{k\sin A}\cdot\sin\frac{A}{2}$$

$$= \frac{2\sin\left(\frac{B+C}{2}\right)\cdot\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}\cdot\sin\frac{A}{2}$$

$$= \frac{\sin\left(\frac{B+C}{2}\right)\cdot\cos\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$$

$$= \frac{\cos\frac{A}{2}\cdot\cos\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}} \qquad \textbf{(Theorem 6·17)}$$

$$= \cos\left(\frac{B-C}{2}\right) = \text{L. H. S.}$$

Ex. 5. In any $\triangle ABC$ prove that

$$4\left[bc\cdot\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cdot\cos^2\frac{C}{2}\right]=(a+b+c)^2$$

Ans. L.H S. $= 4\left[bc\left(\frac{1+\cos A}{2}\right)+ca\left(\frac{1+\cos B}{2}\right)+ab\left(\frac{1+\cos C}{2}\right)\right]$

$$=2\,[bc+bc\cos A+ca+ca\cos B+ab+ab\cos C]$$

$$=2\left[bc+bc\cdot\frac{b^2+c^2-a^2}{2bc}+ca+ca\cdot\frac{c^2+a^2-b^2}{2ca}+ab+ab\cdot\frac{a^2+b^2-c^2}{2ab}\right]$$

$$=2bc+2ca+2ab+(b^2+c^2-a^2)+(c^2+a^2-b^2)+(a^2+b^2-c^2)$$

$$=a^2+b^2+c^2+2ab+2bc+2ca$$

$$=(a+b+c)^2=\text{R. H. S.}$$

Ex. 6. In $\triangle ABC$ prove that

$$a\cdot\sin(B-C)+b\sin(C-A)+c\cdot\sin(A-B)=0$$

Ans. Let $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \quad\therefore\ a=k\sin A$ etc.

$$\therefore\ a\sin(B-C)=k\sin A\cdot\sin(B-C)$$

$$=k\cdot\sin(B+C)\cdot\sin(B-C)$$

$$[\because\ \sin A=\sin(B+C)]$$

$$=\frac{k}{2}\,2\sin(B+C)\cdot\sin(B-C)$$

$$=\frac{k}{2}\,[\cos(2C)-\cos(2B)]$$

Similarly $b\sin(C-A)=\left[\frac{k}{2}(\cos 2A-\cos 2C)\right]$

and $c\sin(A-B)=\left[\frac{k}{2}(\cos 2B-\cos 2A)\right]$

$$\therefore\ \text{L.H.S.}=\frac{k}{2}[\cos 2C-\cos 2B]+\frac{k}{2}[\cos 2A-\cos 2C]+\frac{k}{2}[\cos 2B-\cos 2A]$$

$$= \frac{k}{2}[\cos 2C - \cos 2B + \cos 2A - \cos 2C + \cos 2B - \cos 2A]$$

$$= \frac{k}{2}(0) = 0 = \text{R.H.S.}$$

Ex. 7. If $\triangle ABC$ is right angled at C, prove that

$$b = c - c \sin A \cdot \tan \frac{A}{2}$$

Ans. $\text{R.H.S.} = c - c \sin A \tan \frac{A}{2}$

$$= c - c\left(2 \sin \frac{A}{2} \cdot \cos \frac{A}{2}\right) \cdot \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$$

$$= c - c\left(2 \sin^2 \frac{A}{2}\right)$$

$$= c - c(1 - \cos A)$$

$$= c - c + c \cos A$$

$$= c \cos\left(\frac{\pi}{2} - B\right) \qquad \because A + B = \frac{\pi}{2}$$

$$= c \sin B$$

$$= b \sin C = b \sin \frac{\pi}{2} = b(1) = b = \text{R.H.S.}$$

Ex. 8. In $\triangle ABC$, $a = 2$ cms., $b = 3$ cms and $c = \sqrt{7}$ cms. Find the cosine of the greatest angle of the triangle. Hence show that the triangle ABC is acute angled.

Ans. Since $2 = \sqrt{4}$ and $3 = \sqrt{9}$ $\quad \therefore 2 < \sqrt{7} < 3$

$\therefore b = 3$ cms. is the longest side.

$\therefore \angle B$ is the greatest angle. So we will find cos B.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{(2)^2 + (\sqrt{7})^2 - 3^2}{2(2)(\sqrt{7})} = \frac{4 + 7 - 9}{4\sqrt{7}} = \frac{1}{2\sqrt{7}} > 0.$$

Since $\cos B > 0$, $\quad 0 < B < \frac{\pi}{2}$.

But B is the greatest angle. $\therefore$ ABC is acute angled.

Exercise 7.2 :

(A) In any $\triangle ABC$, prove the following :

(1) $a^2 \sin 2C + c^2 \sin 2A = 2ac \sin B$.

(2) $a^2 \sin (B - C) = (b^2 - c^2) \sin A$.

(3) $c^2 = (a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2}$.

(4) $\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ac} = \frac{\cos C}{c} + \frac{c}{ab}$.

(5) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$.

(6) $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C = \frac{8\Delta^2}{abc}$,
where Δ = Area of $\triangle ABC$.

(7) $a\left(\cos B + \cot \frac{A}{2} \cdot \sin B\right) = b + c$.

(8) If $2b = a + c$, then $3 \tan \frac{A}{2} \cdot \tan \frac{C}{2} = 1$.

(9) $\frac{\cos^2\left(\frac{B-C}{2}\right)}{(b+c)^2} + \frac{\sin^2\left(\frac{B-C}{2}\right)}{(b-c)^2} = \frac{1}{a^2}$.

(10) $\frac{b^2-c^2}{a} \cos A + \frac{c^2-a^2}{b} \cos B + \frac{a^2-b^2}{c} \cos C = 0$.

(11) $a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B) = 0$.

(12) If $C = 2B$, then $c^2 = ab + b^2$.

(B) (1) If $a \cos A = b \cos B$ then prove : either $A = B$ or $C = \frac{\pi}{2}$.

(2) In $\triangle ABC$ if $b = 3$ cms, $c = 2$ cms. and $m\angle A = \frac{2\pi}{3}$, find a and the area of $\triangle ABC$.

(3) In $\triangle ABC$, if $a = 3$ cms., $c = 4$ cms. and $m\angle B = \frac{\pi}{6}$, find the area of $\triangle ABC$. If a and c are unchanged, but $m\angle B$ is increased to $\frac{5\pi}{6}$, will the area change ? Why ?

(4) One side of a triangle is double of the other and the angles opposite these sides differ by $\frac{\pi}{3}$. Show that the triangle is right angled.

§ 7·24. **Half angle formulae :**

In this article, we shall prove formulae for the sine, cosine and tangent of the angles $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ in terms of s, the semi-perimeter of the triangle $\left(s = \frac{a+b+c}{2}\right)$.

We shall also prove a formula for the area of the triangle in terms of s.

Formulae : In triangle ABC, prove that

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$,

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$,

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ where $2s = a+b+c$.

Proof : (i) We know that $\cos A = 1 - 2\sin^2\frac{A}{2}$

$\therefore \quad 2\sin^2\frac{A}{2} = 1 - \cos A$

$= 1 - \frac{b^2+c^2-a^2}{2bc}$ [Using cosine rule]

$= \frac{2bc-b^2-c^2+a^2}{2bc} = \frac{a^2-(b^2-2bc+c^2)}{2bc}$

$= \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$

Let $2s$ be the perimeter of the triangle ABC.

$\therefore \quad 2s = a+b+c.$

Now $a+b-c = (a+b+c)-2c = 2s-2c$

and $a-b+c = (a+b+c)-2b = 2s-2b$

$\therefore \quad 2\sin^2\frac{A}{2} = \frac{(2s-2c)(2s-2b)}{2bc} = \frac{2(s-c)\cdot 2(s-b)}{2bc}$

$\therefore \quad 2\sin^2\frac{A}{2} = \frac{2(s-b)(s-c)}{bc}$

$\therefore \quad \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$

$$\therefore \quad \sin\frac{A}{2} = \pm\sqrt{\frac{(s-b)(s-c)}{bc}}$$

Now, since $m\angle A < 180°$, $m\angle\frac{A}{2} < 90°$, we reject the negative sign.

$$\therefore \quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

Similarly, we can obtain

$$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \quad \text{and} \quad \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

(ii) Let us obtain the formula for $\cos\frac{A}{2}$

Since $\cos A = 2\cos^2\frac{A}{2} - 1$

$$\therefore \quad 2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{2s\cdot(2s-2a)}{2bc} = \frac{2s\cdot 2(s-a)}{2bc} = \frac{2s\cdot(s-a)}{bc}$$

$$\therefore \quad 2\cos^2\frac{A}{2} = \frac{2s(s-a)}{bc} \qquad \therefore \quad \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$$

$$\therefore \quad \cos\frac{A}{2} = \pm\sqrt{\frac{s(s-a)}{bc}}$$

As in (i), $m\angle\frac{A}{2} < 90°$. So, rejecting the negative sign, we get

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

Similarly we can show that

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}, \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

(iii) Now $\tan\frac{A}{2}=\sin\frac{A}{2}\div\cos\frac{A}{2}$

$$=\sqrt{\frac{(s-b)(s-c)}{bc}}\div\sqrt{\frac{s(s-a)}{bc}}$$

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Formula for the area: In $\triangle$ ABC, prove that $\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$, and hence deduce that the area of the triangle is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where $2s=a+b+c$.

Proof: We know that

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}\text{ (as proved above)}$$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Similarly $\sin B=\frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$

and $\sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$.

Next, we know that area of $\triangle ABC=\frac{1}{2}bc\sin A$

(Ex. 2, P 130)

Substituting the value of sin A in it, we get

area of $\triangle ABC=\frac{1}{2}bc\sin A$

$$=\frac{1}{2}bc\cdot\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{s(s-a)(s-b)(s-c)}.$$

§ 7·25. Circumradius and Inradius of a triangle:

Definition: (1) If ABC is a triangle and a circle is drawn to pass through A, B, C; it is called the *circumcircle* of the triangle and the radius of this circle is called the *circumradius* of the triangle. The circumradius will be denoted by R.

(2) If ABC is a triangle and a circle is drawn inside the triangle such that *each* side of the triangle touches the circle, the circle is called the *incircle* of the triangle and its radius is called the *inradius* of the triangle. The inradius will be denoted by r.

Formulae: In $\triangle$ ABC, prove that

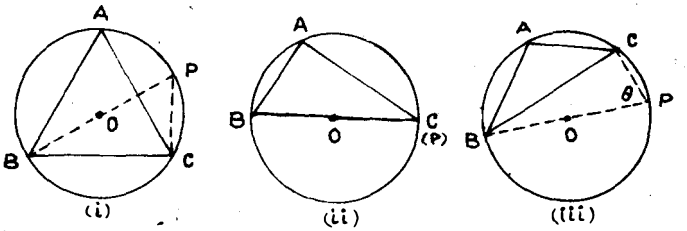
(i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

where R is the circumradius

(ii) Circumradius $R = \frac{abc}{4\triangle}$ where $\triangle$ = area of $\triangle$ ABC.

(iii) Inradius $r = \frac{\triangle}{s}$ where $s = \frac{a+b+c}{2}$.

Proof: (i) Let O be the circumcentre of $\triangle$ABC and R be its circumradius. Let BO meet the circumference at P. Let $m \angle bPC = \theta$.



In fig. (i), $\angle A$ is acute and $m \angle A = m \angle P$.

$\therefore \quad \sin A = \sin P = \frac{l(BC)}{l(BP)} = \frac{a}{2R}$.

In fig. (ii), $m\angle A = 90° \quad \therefore \sin A = 1 = \frac{a}{2R} \quad (\because a = 2R)$

In fig. (iii) $\angle A$ is obtuse, and $m \angle A + m \angle P = 180°$

$\therefore \quad m \angle A = 180° - m \angle P = 180° - \theta$

$\therefore \quad \sin A = \sin (180° - \theta) = \sin \theta = \frac{l(BC)}{l(BP)} = \frac{a}{2R}$.

Hence in all cases, $\sin A = \frac{a}{2R}$. $\quad \therefore \quad \frac{a}{\sin A} = 2R$.

Similarly we can show that $\frac{b}{\sin B} = 2R$, $\frac{c}{\sin C} = 2R$.

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin C} = \frac{c}{\sin C} = 2R \qquad \ldots (I)$

(*Note* : This result also proves the Sine Rule)

(ii) As proved in (I), $\frac{a}{\sin A}=2R \quad \therefore \sin A=\frac{a}{2R}$ .. (i)

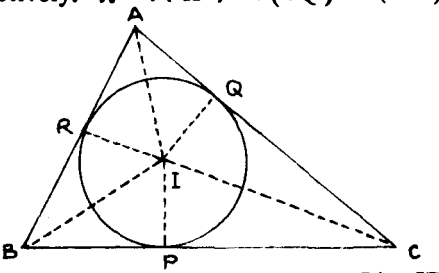
We know that, if $\triangle$ is the area of $\triangle$ ABC then

$\triangle=\frac{1}{2}bc\sin A \quad \therefore \sin A=\frac{2\triangle}{bc}$ .. (ii)

$\therefore$ From (i) and (ii), $\frac{2\triangle}{bc}=\frac{a}{2R}$

$\therefore 4R\triangle=abc$ i.e. $R=\frac{abc}{4\triangle}$ .. (II)

(iii) Let I be the centre and r be the radius of the incircle of $\triangle$ ABC. Let this circle touch the side BC, CA and AB at P, Q and R respectively. $\therefore l(IP)=l(IQ)=l(IR)=r$.



Let $\triangle$ be the area of $\triangle$ ABC. Join IA, IB, IC. Let $\triangle_1$, $\triangle_2$, $\triangle_3$ be areas of $\triangle$ IBC, $\triangle$ ICA and $\triangle$ IAB respectively.

$\therefore \triangle=\triangle_1+\triangle_2+\triangle_3$

$\therefore \triangle=\frac{1}{2}l(BC)\cdot l(IP)+\frac{1}{2}l(CA)\cdot l(IQ)+\frac{1}{2}l(AB)\cdot l(IR)$.

$=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=r\frac{(a+b+c)}{2}=rs$ where $s=\frac{a+b+c}{2}$.

$\therefore r=\frac{\triangle}{s}$ as required.

§ 7·26. Examples :

Ex 1. In $\triangle$ ABC. if $a=6$ cms. and $m\angle A=\frac{\pi}{6}$, find the circumradius R.

Ans. We know that

$2R=\frac{a}{\sin A}$. $\therefore 2R=\frac{6}{\sin\frac{\pi}{6}}=\frac{6}{\left(\frac{1}{2}\right)}=12$.

$\therefore 2R=12$. $\therefore R=6$ cms.

Ex. 2. In $\triangle ABC$, prove that $\tan\frac{A}{2}=\frac{r}{s-a}$.

Ans. We know that

$$r=\frac{\Delta}{s}=\frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore \text{ R. H. S.}=\frac{r}{s-a}=\frac{1}{s-a}\cdot\left\{\frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}\right\}$$

$$=\frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}=\tan\frac{A}{2}=\text{L. H. S.}$$

Ex. 3. In $\triangle ABC$, $a=18$ cms., $b=24$ cms. and $c=30$ cms. Find sin A.

Ans. In this case $2s=18+24+30=72$ cms. $\therefore s=36$ cms.

Now $\sin A=\frac{2}{bc}\cdot\sqrt{s(s-a)(s-b)(s-c)}$

$$=\frac{2}{(24)(30)}\sqrt{36(36-18)(36-24)(36-30)}$$

$$=\frac{1}{360}\sqrt{36(18)(12)(6)}=\frac{1}{360}\cdot(6^3)=\frac{6}{10}=\frac{3}{5}.$$

Exercise 7.3 :

(1) In $\triangle ABC$, if $a=25$ cms., $b=52$ cms. and $c=63$ cms., show that $\tan\frac{C}{2}=\frac{9}{7}$.

(2) In $\triangle ABC$, if $a=125$ cms., $b=123$ cms, and $c=62$ cms., show that $\sin\frac{B}{2}=\frac{3}{5}$.

(3) In $\triangle ABC$, if $a=13$ cms., $b=14$ cms. and $c=15$ cms., show that $\sin A=4/5$.

(4) In $\triangle ABC$, prove that $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\Delta}$.

(5) In $\triangle ABC$, if $B=90°$, show that $\tan\frac{A}{2}=\sqrt{\frac{b-c}{b+c}}$.

(6) In $\triangle ABC$, prove that $Rr(\sin A+\sin B+\sin C)=\Delta$.

(7) In $\triangle ABC$, show that $\cot\frac{A}{2}-\cot\frac{B}{2}=\frac{(b-a)s}{\Delta}$.

(8) In $\triangle ABC$, show that $\tan\frac{C}{2}=\frac{r}{s-c}$.

§ 7·3. Heights and distances :

With the help of the formulae of trigonometry, we can find distances between points, heights of objects etc. without actually measuring them.

§ 7·31. Angles of elevation and depression :

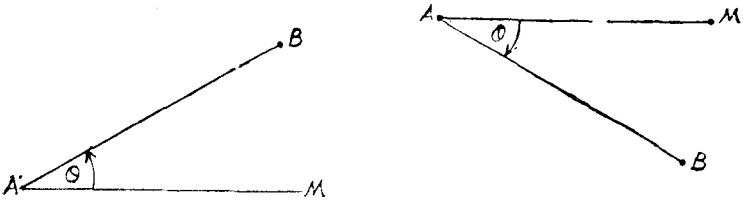


Fig. 7·9

Consider two points A and B which are not on the same level. Then either A is on a higher level than B or on a lower level than B as shown in fig. 7·9. Draw the horizontal line AM through A.

If B is on a higher level than A, then the angle $MAB = \alpha$ (shown in fig. 7·9) is called the angle of elevation of B with respect to A (or angle of elevation of B as seen from A).

If A is on a higher level than B then the angle $MAB = \theta$ (shown in fig. 7·9) is called the angle of depression of B with respect to A (or angle of depression of B as seen from A).

In short, we can say that if our eye is at A looking horizontally ahead then the angle of elevation (or depression) of B is the angle through which the eye has to be raised (or lowered) if we want to look at B.

Using these terms and the formulae of trigonometry, we can solve a number of examples.

§ 7·32. Examples :

Ex. 1. A pole is fixed on the top of a building. The angles of elevation of the top of the pole and the top of the building are 60° and 45° respectively as observed from a point on the ground at a horizontal distance of 60 feet from the building. Find the length of the pole.

Ans. Let AB be the building and BC be the pole. Let P be the point on the ground such that $l(PA) = 60'$.

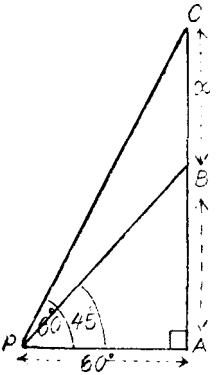


Fig. 7·10

$\therefore$ $m\angle APB = 45°$ and $m\angle APC = 60°$.

Let $l(BC) = x'$. Obviously, in right angled triangle APB,

$m\angle APB = m\angle ABP = 45°$

$\therefore$ $l(AB) = l(AP) = 60'$

$\therefore$ $l(AC) = (60 + x)'$.

In the right triangle APC,

$$\tan 60° = \frac{l(AC)}{l(AP)} \quad \therefore \quad \sqrt{3} = \frac{60+x}{60}$$

$$\therefore \quad \sqrt{3} = 1 + \frac{x}{60} \quad \therefore \quad \sqrt{3} - 1 = \frac{x}{60}$$

$$\therefore \quad x = 60(\sqrt{3} - 1)'.$$

$\therefore$ The height of the pole $= 60(\sqrt{3} - 1)$ feet.

Ex. 2. A man is walking towards a tower with a uniform velocity. If the angle of the elevation of the top of the tower changes, in three minutes, from 30° to 60°, find the velocity of the man given that the height of the tower is $45\sqrt{3}'$ and the angles of elevation are measured from the ground.

Ans. Let PQ be the tower of height $45\sqrt{3}'$. Let the man walk, in three minutes, from A to B.

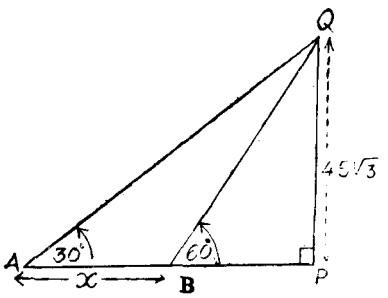


Fig.7.11

$\therefore$ $m\angle PAQ = 30°$ and $m\angle PBQ = 60°$.

Let $l(AB) = x'$ and $l(BP) = y'$

$\therefore$ $l(AP) = (x + y)'$.

Now in right triangle BPQ,

$$\tan 60° = \frac{l(PQ)}{l(BP)}$$

$$\therefore \quad \sqrt{3} = \frac{45\sqrt{3}}{y}, \quad \therefore \quad y = 45.$$

In right triangle APQ, $\tan 30° = \dfrac{l(PQ)}{l(AP)}$

$$\frac{1}{\sqrt{3}} = \frac{45\sqrt{3}}{x+y} \quad \therefore \quad x + y = 45(3) = 135'$$

But $y = 45'$ $\therefore$ $x = 90'$. Hence the man walks 90′ in 3 minutes.

$\therefore$ His speed is 30 ft./min. or 0·5 ft./sec.

Ex. 5. A tower is 50 metres high. Its shadow is x metres longer when the Sun's altitude is 52° than when it is 32°. **Find** x. (Given tan 58° = 1·6003 and tan 38° = 0·7813).

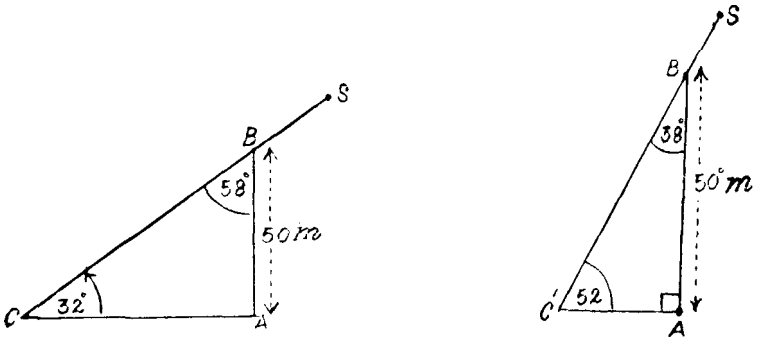


Fig. 7·12

Ans. Let AB be the tower and S be the Sun.

Let AC and AC′ be the shadows when the Sun's altitude is 32° and 52° respectively. $\therefore\ l(AC) - l(AC') = x$ metres.

Now $m\angle ACB = 32°$ and $m\angle AC'B = 52°$

$\therefore\ m\angle ABC = 58°$ and $m\angle ABC' = 38°$

In right triangle ABC, $\tan 58° = \dfrac{l(AC)}{l(AB)} = \dfrac{l(AC)}{50}$

$\therefore\ l(AC) = 50 \tan 58°$ ..... (i)

In right triangle ABC′, $\tan 38° = \dfrac{l(AC')}{l(AB)} = \dfrac{l(AC')}{50}$

$\therefore\ l(AC') = 50 \tan 38°$ ..... (ii)

$\therefore\ x = l(AC) - l(AC') = 50 \tan 58° - 50 \tan 38°$

[by (i) and (ii)]

$\therefore\ x = 50 (\tan 58° - \tan 38°)$

$= 50 (1·6003 - 0·7813) = 50 (0·8190) = 40·95$

$\therefore\ x = 40·95$ metres.

Note: The values of the tangents of 58° and 38°, if not given, can be found using trigonometric tables.

Exercise 7·2 :

(1) A tower stands on the top of a hill, 300′ away from an observer on the ground. He obtains the angles of elevation of the top and bottom of the tower as 75° and 60° respectively. Find the height of the hill and the tower.

(2) A person walking along a straight road observes that, at two consecutive kilometre stones the angles of elevation of a hill in front of him are 30° and 75° respectively. Find the height of the hill.

(3) A tree casts a shadow when the sun is 30° above the horizon. When it rises 45° degrees above the horizon, the length of the shadow reduces by 20′. Find the height of the tree.

(4) The pilot of an aeroplane observes two towers on a straight road two miles apart, the plane being vertically above the line joining the towers. If the angles of depression of the tops of the towers are 60° and 30°, find the height of the aeroplane, given that height of each tower is $\frac{1}{2}$ mile.

(5) From the foot of a column, the angle of elevation of the top of a tower is 45° and from the top of the column, the angle of depression of the bottom of the tower is 30°. A man walks 10′ from the bottom of the column towards the tower and notices the angle of elevation of the top of the tower to be 60°. Find the height of the column.

(6) A cloud is observed above a river from opposite banks and the angles of elevation of the cloud are 45° and 30°, the cloud being vertically above the line joining the points of observation. If the river is 440′ wide, find the height of the cloud.

(7) Two lamp posts are of equal height. A person standing midway between the line joining their bases observes the elevation of the top of either post to be 30°. After walking 40′ towards one of them, he observes the angle of elevation of its top to be 60°. Find the heights of the posts and the distance between them.

(8) From the deck of a ship, the elevation of the top of a cliff is 50° and from the top of a mast 50′ above the deck, the elevation is 35°. Show that the height of the cliff is approximately 121′ above the deck level.

(9) Two balloons reaching the same height of 200′ observe a green patch on the ground lying between them. If the angles of depression are 36° and 54°, find the distance between the balloons.

(10) The angle of elevation of a cloud from a point 100′ above a lake is 30° and the angle of depression of its reflection in the lake is 60° from the same point. Find the height of the cloud.

CHAPTER 8

The Straight Line

The straight line is a very important locus in coordinate geometry. A straight line can be defined as the locus of a point such that if any two points of this locus are joined, they define a unique 'direction'. To understand the concept of 'direction', we shall define the inclination of a line and the slope of a line.

§ 8·1. Inclination of a Line :

The inclination of a line depends on the angle made by a line with the x axis. Now if a line cuts the x axis at a point A as shown in fig. 8·1, then the angle can be measured from AX in the anticlockwise direction (shown by θ) and also from AO in the clockwise direction (shown by α). As a convention, the angle made by a line with the x axis is always measured from AX in the anti-clockwise direction where A is the point where the line meets the x axis. The direction AX of the x axis is referred to as : 'the positive direction of x axis.'

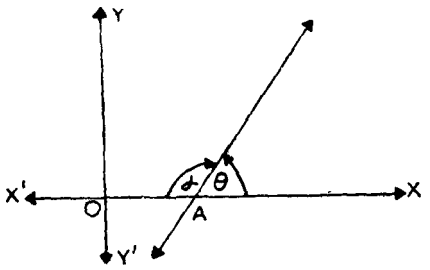


Fig. 8·1

Now consider fig. 8·2. Three lines AB, AC and RS are shown in it. The line AB meets the x axis at D and the line AC meets the x axis at E, A being on the y axis. Suppose $l[OA] = l[OD] = l[OE]$ where O is the origin. Hence, the measure of angle made by AB with the positive direction of x axis is $m\angle XDB = 45°$. Similarly, the measure of angle

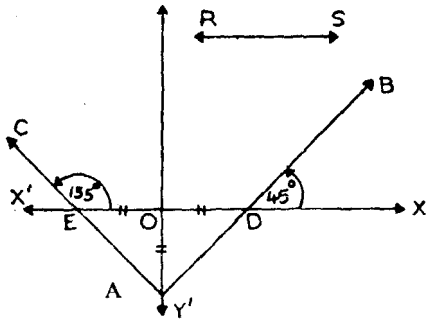


Fig 8·2

made by the line AC with the positive direction of x axis is $m\angle XEC = 135°$.

Note that when a straight line intersects the x axis, some portion of it will be above the x axis and hence the angle θ made by it with the positive direction of x axis must satisfy $0 \leq \theta < \pi$ or $0° \leq \theta < 180°$, e. g. for the line AB the portion DB is above x axis and hence the measure of the angle is measured as $m\angle XDB$ (and not as $m\angle XDA$).

If a straight line does not intersect the x axis, it is parallel to the x axis and the angle made by it with the x axis is defined as $0°$ or 0^c, e. g. the line RS in fig. 8·2 is parallel to the x axis and hence the angle made by it with the x axis is $0°$ or 0^c. Hence we can give the following definition :

Definition : Inclination of a line :

If a straight line intersects the x axis, the *inclination* of the line is defined as the measure of the smallest non-negative angle which the line makes with the positive direction of the x axis. If a straight line is parallel to the x axis, its *inclination* is defined to be *zero.*

$\therefore$ If θ is the inclination of a line, then $0 \leq \theta < 180°$ or $0 \leq \theta < \pi$.

e. g. In fig. 8·3, the line AB has inclination $45°$ even though the angle made by it with the x axis can be considered as $45°$, $360° + 45°$, $2(360° + 45°,)$.... or even $-135°$, $-360° - 135°$, ...This is because the inclination is the *smallest non-negative* angle made with the positive direction of x axis and hence it is $45°$.

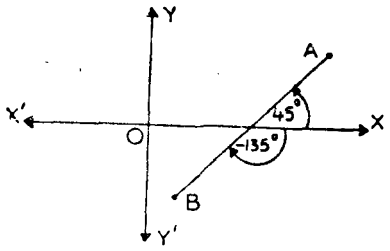


Fig. 8·3

§ 8·11. Slope of a line :

Having defined the inclination of a line, we are now in a position to define the *slope* of a line, in terms of its inclination. You have already learnt that the slope of a line is equal to the slope of any line segment contained in it. We shall give the definition of slope in terms of the inclination of a line and then prove its equivalence with the definition of slope you have learnt earlier.

Definition : Slope of a line : If the inclination of a line is θ and $\theta \neq \frac{\pi}{2}$ then the *slope* of the line is defined to be ' tan θ '.

Note : For a vertical line, i. e. a line parallel to y axis, the inclination of the line is $\frac{\pi}{2}$ and hence its slope is not defined Hence the slope of a line, parallel to y axis, does not exist or a line parallel to y axis has *no slope*. The ideas of '*no slope*' and '*zero slope*' are different. If a line is parallel to the y axis, it has no slope but if a line is parallel to x axis, its inclination is zero and hence its slope is zero. ($\because$ tan $\theta = \tan 0 = 0$). Hence if the slope of a line is denoted by the letter 'm', then for a line parallel to x axis, $m = 0$ and for a line parallel to y axis, m does not exist.

§ 8·12. Intercepts of a line :

If a straight line intersects the x axis at A, then the x co-ordinate of A is called the *intercept* made by the line on the x axis (or, in short, the x *intercept* of the line).

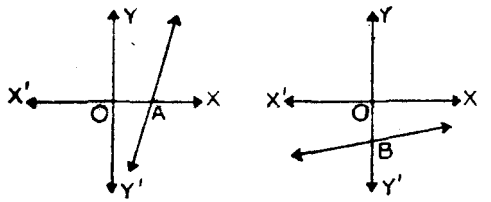


Fig. 8·4

Similarly, if a straight line intersects the y axis at B, then the y coordinate of B is called the *intercept* made by the line on the y axis (or, in short, the y-*intercept* of the line).

Note : The intercept on the x axis or the y axis made by a straight line can be positive, negative or zero depending on the coordinates of the point where the line meets the axis concerned.

§ 8·13. Parallel and Perpendicular Lines :

(I) Parallel lines : Consider two lines which are not parallel to y axis. If these lines are parallel to each other as shown in fig. 8·5, then they have the same inclination and hence their slopes are equal.

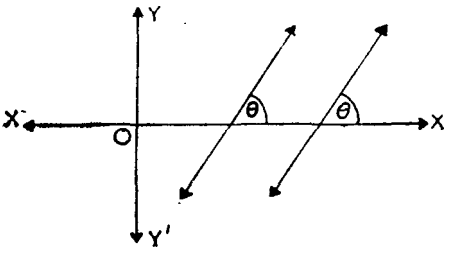


Fig. 8·5

Conversely, if the slopes of two lines are equal,

they must be parallel. Since, if the slopes of two lines are $\tan\theta_1$ and $\tan\theta_2$ such that $\tan\theta_1 = \tan\theta_2$ then $\theta_1 = \theta_2$ since $0 \leqslant \theta_1 < \pi$, $0 < \theta_2 < \pi$. Hence the lines have the same inclination and are parallel.

Further, in case two lines are parallel to the y axis, they are parallel to each other and both of them have no slope. Conversely, if the slopes of two lines do not exist, both must be parallel to the y axis and hence parallel to each other.

Hence we conclude : " Two lines whose slopes are m_1 and m_2 are parallel to each other, if and only if, either (i) $m_1 = m_2$ or (ii) both m_1 and m_2 do not exist."

(II) **Perpendicular lines :** Let two lines having inclinations

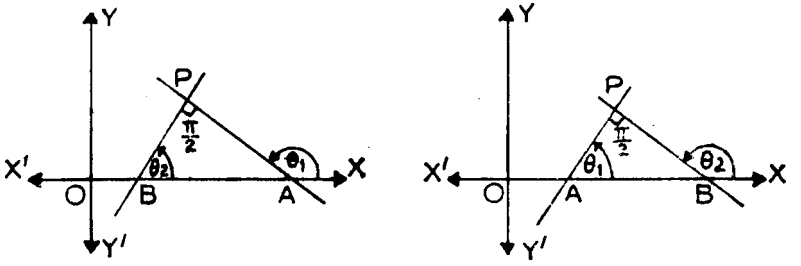


Fig. 8·6

θ_1 and $\theta_2 \left(\theta_1 \neq \frac{\pi}{2}, \theta_2 \neq \frac{\pi}{2}\right)$ be perpendicular to each other. Then either $\theta_1 - \theta_2 = \frac{\pi}{2}$ or $\theta_2 - \theta_1 = \frac{\pi}{2}$ as shown in fig. 8·6. Hence either $\theta_1 = \frac{\pi}{2} + \theta_2$ or $\theta_2 = \frac{\pi}{2} + \theta_1$. If $\theta_1 = \frac{\pi}{2} + \theta_2$, then $\tan\theta_1 = \tan\left(\frac{\pi}{2} + \theta_2\right) = -\cot\theta_2 = \frac{1}{\tan\theta_2}$. If $\theta_2 = \frac{\pi}{2} + \theta_1$, then $\tan\theta_2 = \tan\left(\frac{\pi}{2} + \theta_1\right) = -\cot\theta_1 = -\frac{1}{\tan\theta_1}$. In either case $\tan\theta_1 \tan\theta_2 = -1$. Hence if the slopes of these two perpendicular lines are denoted by m_1 and m_2 then $m_1 m_2 = -1$.

Conversely, if two lines have slopes m_1 and m_2 such that $m_1 m_2 = -1$, then they are perpendicular to each other.

Now consider two lines, one parallel to x axis and the other parallel to y axis. These are perpendicular to each other where one has zero slope and the slope of the other does not exist.

Hence we conclude : " Two lines whose slopes are m_1 and m_2 are perpendicular to each other, if and only if, either (i) $m_1 m_2 = -1$ or (ii) $m_1 = 0$ and m_2 does not exist. "

§ 8·14. Slope of a line joining two points :

You have already learnt that the slope of a line is the slope of any line segment contained in it. We have now defined slope of a line as $\tan\theta$ where $\theta \left(\neq \frac{\pi}{2}\right)$ is the inclination of the line. We shall now prove the equivalence of the two definitions,

(I) Slope of seg OP where O $\equiv$ (0, 0) and P $\equiv$ (x, y), $x \neq 0$:

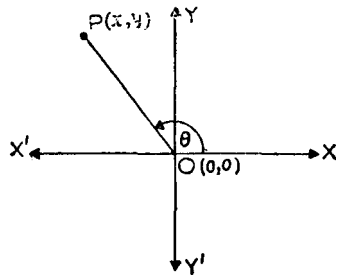


Fig. 8·7

Let the inclination of seg OP be θ. Since $x \neq 0$, P does not lie on the y axis. $\therefore \theta \neq \frac{\pi}{2}$.

$\therefore$ Slope of seg OP $= \tan\theta$

But since $m\angle XOP = \theta$ which is in standard position, $\tan\theta = \frac{y}{x}$ by the definition of trigonometric ratios.

$\therefore$ Slope of OP $= \tan\theta = \frac{y}{x}$.

$\therefore$ Slope of the line joining O(0, 0) and P(x, y) is $\frac{y}{x}$ .. **(I)**

(II) Slope of seg AB where A $\equiv$ (x_1, y_1) and B $\equiv$ (x_2, y_2), $x_1 \neq x_2$:

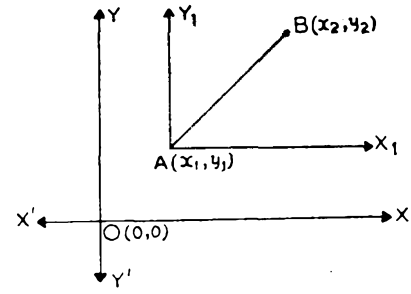


Fig. 8·8

Let A(x_1, y_1) and B (x_2, y_2) be any two points. Since $x_1 \neq x_2$, seg AB is not parallel to y axis.

Now shift the origin to A (x_1, y_1), keeping the new axes parallel to the old axes. Hence the new coordinates of A and P with respect to these new axes are (0, 0) and $(x_2 - x_1, y_2 - y_1)$. Hence the slope of the line joining (0, 0) and

$(x_2 - x_1, y_2 - y_1)$ is $\frac{y_2 - y_1}{x_2 - x_1}$ by (I) given above.

$\therefore$ Slope of seg AB $= \frac{y_2 - y_1}{x_2 - x_1}$ .... (II)

This is precisely the slope of the line as you have learnt earlier. Hence the two definitions of slope are equivalent.

§ 8·15. Examples :

Ex. 1. Find the slope of the line joining A (2, 3) and B (5, 7).

Ans. Slope of seg AB $= \frac{3-7}{2-5} = \frac{-4}{-3} = \frac{4}{3}$.

Ex. 2. Show that the points A (3, 1); B (-1, 9) and C (4, -1) are collinear.

Ans. Slope of seg AB $= \frac{1-9}{3-(-1)} = \frac{-8}{4} = -2$,

and slope of seg BC $= \frac{9-(-1)}{-1-4} = \frac{10}{-5} = -2$.

Since slope of seg AB = slope of seg BC and B is common to both lines, seg AB and seg BC belong to the same line. Hence, A, B, C are collinear.

Ex. 3. Show that the points A (2, 2); B (3, 4) and C (4, 1) are the vertices of a right angled triangle.

Ans. Here slope of seg AB $= \frac{2-4}{2-3} = 2$,

slope of seg BC $= \frac{4-1}{3-4} = -3$

and slope of seg AC $= \frac{2-1}{2-4} = -\frac{1}{2}$

$\therefore$ (slope of seg AB)·(slope of seg AC) $= 2 \cdot \left(-\frac{1}{2}\right) = -1$.

$\therefore$ seg AB is perpendicular to seg AC.

$\therefore$ ABC is a triangle right angled at A.

Ex. 4. By using slopes, show that the points A (2, 5); B (3, -2); C (-4, 1) and D (-5, 8) are the vertices of a parallelogram.

Ans. Here slope of seg AB $= \frac{5-(-2)}{2-3} = -7$;

slope of seg BC $= \frac{-2-1}{3-(-4)} = -\frac{3}{7}$;

$$\text{slope of seg CD} = \frac{1-8}{-4-(-5)} = -7;$$

$$\text{and slope of seg DA} = \frac{8-5}{-5-2} = -\frac{3}{7}.$$

Since slope of seg AB = slope of seg CD, seg AB ∥ seg CD and since slope of seg BC = slope of seg DA, seg BC ∥ seg DA.

∴ ABCD is a aparallelogrom.

Exercise 8·1 :

(1) Find the slope of a line which makes

(a) an angle of 45° with the positive direction of x axis,

(b) an angle of 30° with the positive direction of y axis,

(c) equal positive intercepts on the axes,

(d) equal negative intercepts on the axes,

(e) equal intercepts on the axes, which are opposite in sign.

(2) Fill in the blanks in the following :

(a) If A ≡ (−2, 3) and B ≡ (4, −1), then slope of AB = ——.

(b) If P ≡ (7, 5) and Q ≡ (−3, 2), then slope of PQ = ——.

(c) If x-intercept of a line is 3, the line meets the x axis at (——, ——).

(d) If y-intercept of a line is −7, the line meets the y axis at (——, ——).

(e) If the x and y-intercepts of a line are 4 and −3 respectively, then its slope is ——.

(f) If AB ∥ CD and slope of seg AB = −3, the slope of seg CD = ——.

(g) If PQ ⊥ RS and slope of seg PQ = $-\frac{2}{3}$, the slope of seg RS = ——.

(h) ABCD is a rectangle and slope of seg AB = $\frac{4}{7}$.

∴ Slope of seg BC = ——; Slope of seg CD = ——.

(3) (a) Show that the points (4, 1), (2, −3) and (1, −5) are collinear.

(b) Show that the points (3, 0), (2, 3) and (−1, 12) are collinear.

(c) If (7, a), (−5, 2) and (3, 6) are collinear, find 'a'.

(d) If (3, k) lies on the line joining (−1. 4) and (2, 5) find k.

(4) (a) Show that (12, 8) (−2, 6) (6, 0) form a right angled triangle.

(b) Show that (2, 1), (6, 5) and (4, 7) are the vertices of a right angled triangle.

(c) Prove that (−1, 2) (3, −1) (2, 6) are the vertices of an isoscels right angled triangle.

(d) Find a, if the triangle formed by A (4, 3), B (6, −2) and C (a, −3) is right angled at A.

(5) In a triangle ABC, A is (−2, 3), B (2, −1) and C (4, 0) seg AD and BE are the medians, seg AM and BN are the altitudes Find the slopes of segs, AB, BC, CD, AD, BE, AM, BN and the perpendicular bisectors of seg BC and seg AC.

(6) (1) Show that the points (3, 0), (0, 2), (−6, 8) and (9, −2) are the vertices of a trapezium.

(2) Show that (8, 3), (2, −1), (0, 1) and (6, 5) are the vertices of a parallelogram.

(3) Show that (3, 2), (0, 4), (−4, −2), (−1, −4) are the vertices of a rectangle.

§ 8·2. Equation of a line :

We shall now study the equation of a line in different forms:

§ 8·21. Lines parallel to the axes :

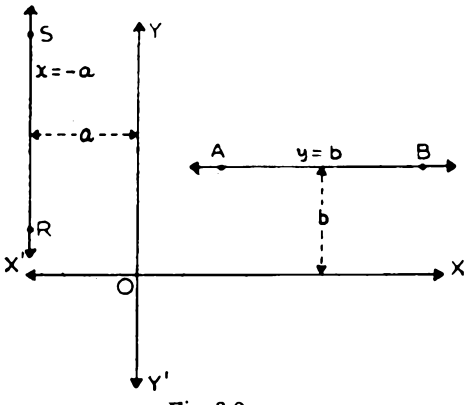


Fig. 8·9

Consider a seg AB, parallel to the x axis, at a distance 'b' units above it. As every point on it has the same y coordinate viz. b, every point of the line satisfies the equation $y = b$. Hence the equation of the line is $y = b$.

Now consider seg RS parallel to y axis, at a distance 'a' units to the left of it. As every point on it has the same x coordinate, viz. $-a$, every point of the line satisfies the equation $x = -a$. Hence the equation of the line is $x = -a$. Hence we conclude:

"If a line is parallel to the x axis and at a distance k units above it, its equation is $y = k$. If it is parallel to the x axis and at a distance k units below it, its equatinn is $y = -k$."

Similarly, "if a line is parallel to the y axis and at a distance

k units to the right of it, its equation is $x = k$. If it is parallel to the y axis and at a distance k units to the left of it, its equation is $x = -k$."

Note : (i) Let the lines $x = a$ and $y = b$ meet at P. Since P lies on $x = a$, the x coordinate of P is 'a'. Similarly, the y coordinate of P is 'b'. $\therefore$ P $\equiv (a, b)$. P is called the point of intersection of the lines $x = a$ and $y = b$.

(ii) Since the line $x = a$ is parallel to the y axis and the line $y = b$ is parallel to the x axis, they are perpendicular to each other. Hence, the angle between the lines $x = a$ and $y = b$ is $\frac{\pi}{2}$ or 90°.

(iii) Consider two lines $x = a$ and $x = b$ where $a \neq b$. Since both the lines are parallel to the y axis, they are parallel to each other and the distance between them is $(a - b)$ or $(b - a)$ whichever is positive.

Similarly, the distance between the lines $y = a$ and $y = b$ $(a \neq b)$ is $(a - b)$ or $(b - a)$ whichever is positive.

(iv) Since every point on the x axis has y coordinate zero, the equation of the x axis is $y = 0$. Similarly, the equation of the y axis is $x = 0$.

Exercise 8·2 :

(1) Fill in the blanks :—

(a) Point of intersection of $x = 4$ and $y = -5$ is (———, ———) and the angle between them is ———.

(b) Angle between the lines $4x + 5 = 0$ and $6y - 7 = 0$ is ——— and their point of intersection is (———, ———).

(c) The equation of a line passing through $(-3, 0)$ and $(4, 0)$ is ———.

(d) The equation of a line passing through $(7, -1)$ and $(7, 4)$ is ———.

(e) The equation of the line AB where A is $(3, 4)$ and B is $(3, 0)$ is ———

(f) The distance between the lines $x = 3$ and $x = 7$ is ———.

(g) The distance between the lines $y = 1$ and $y = 3$ is ———.

(2) State, with reasons, whether the following are true or false :

(a) The points $(-3, 0)$, $(0, 0)$ and $(5, 0)$ are collinear.

(b) The points $(0, -3)$, $(0, 0)$ and $(0, 5)$ are collinear.

(c) If A $(4, 5)$, B $(4, 6)$, C $(1, 0)$, D $(1, -7)$ then AB $\parallel$ CD.

(d) Equations of AB and CD are $4x+5=0$ and $3y-7=0$.
$\therefore$ They are perpendicular to each other.

(e) Equation of a line parallel to $x=3$ and passing through (4, 5) is $x=4$.

(f) Distance between the lines $x=4$, and $x=-5$ is 9 units.

(3) (a) Write down the equation of (i) x axis, (ii) y axis.

(b) Write down the equation of a line through the point (3, 5) and (i) parallel to the x axis, (ii) perpendicular to the x axis, (iii) parallel to the y axis, (iv) perpendicular to the y axis.

(4) Write down the equation of the line

(i) parallel to the x axis and at a distance of 3 from it.

(ii) parallel to the y axis and at a distance of 3 from it.

(iii) perpendicular to the x axis and at a distance of 3 from y axis.

(iv) perpendicular to the y axis and at a distance of 3 from the x axis.

(v) parallel to the x axis and at a distance of 5 from the point (2, 3).

(vi) perpendicular to the y axis and at a distance of 5 from the point $(-2, 3)$.

(5) Write down the equation of a line

(i) parallel to the x axis and 2 units below it.

(ii) parallel to the y axis and 5 units to the left of it.

(iii) parallel to the x axis and making an intercept of 3 on the y axis.

(iv) parallel to the y axis and making an intercept of 5 on the x axis.

(v) parallel to the x axis and making an intercept of 5 on the y axis.

(6) Find the angle between the lines $x+2=0$ and $3y-4=0$ and the coordinates of their point of intersection.

(7) A (3, 4) is a point and line AB is parallel to the y axis. If the y coordinate of B is -7, find its x coordinate.

(8) $x=2$, $x=3$, $y=1$, $y=4$ are the sides of a rectangle. Find the co-ordinates of its vertices and the coordinates of the point of intersection of its diagonals.

§8·22. Standard forms for the equation of a line :

We shall now study certain standard forms for the equations of straight lines which are not parallel to the axes in general.

The equation of a straight line can be found under different conditions, e.g. a straight line is determined if we are given, (i) coordinates of two points on the line, or (ii) its slope and the coordinates of one point on it or (iii) the intercepts made by it on the axes etc. In every case, we shall select (x, y) as the coordinates of any point on the line and find an equation involving x and y from the given conditions about the line. Hence we shall obtain a standard formula for the equation in each case. We shall prove five standard forms of this type.

Theorem 8·1 : The slope point form : The equation of a line passing through the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

Proof : Let $P(x, y)$ be any point on the given line. Let $A(x_1, y_1)$ be the given point on it. Since slope of the line is given to be m,

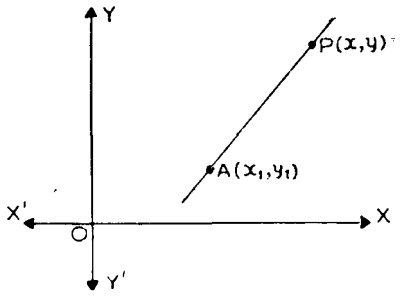


Fig. 8·10

$\therefore$ Slope of seg $AP = m$ .... (i)

But slope of seg AP is $\dfrac{y - y_1}{x - x_1}$ .... (ii)

From (i) and (ii)

$\therefore \dfrac{y - y_1}{x - x_1} = m$

$\therefore y - y_1 = m(x - x_1)$ is the equation of the line.

e.g. if a line has slope 2 and it passes through (1, 4), its equation is :

$$y - 4 = 2(x - 1)$$

$$\therefore \quad y - 4 = 2x - 2$$

$$\therefore \quad 2x - y + 2 = 0$$

Theorem 8·2 : The two points form :

The equations of a straight line passing through the points (x_1, y_1) and (x_2, y_2) where $x_1 \neq x_2$ is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Proof: Let P (x, y) be any point on the given line. Let A (x_1, y_1) and B (x_2, y_2) be the two given points on it.

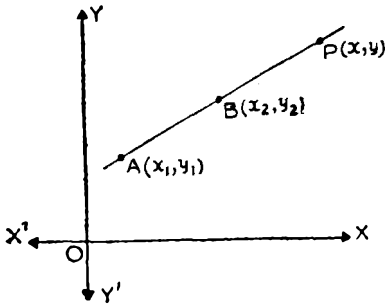


Fig. 8·11

Since A, B and P are collinear.

$\therefore$ Slope of seg AP = slope of seg AB

$$\therefore \frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$$

is the required equation of the line.

e. g. if a straight line passes through the points $(3, 4)$ and $(-1, 2)$, its equation is $\frac{y-4}{x-3}=\frac{4-2}{3-(-1)}$

$$\therefore \frac{y-4}{x-3}=\frac{2}{4} \quad \therefore \quad 2y-8=x-3$$

$\therefore$ $x-2y+5=0$ is the required equation.

§ 8·33. Examples:

Ex. 1. If A $(1,-2)$; B $(-2, 3)$ and C $(2, -5)$ are the vertices of a triangle ABC, find the equation of (i) the altitude AM and (ii) the median BD.

Ans. (i) Slope of BC $= \frac{3-(-5)}{-2-2}=-2$

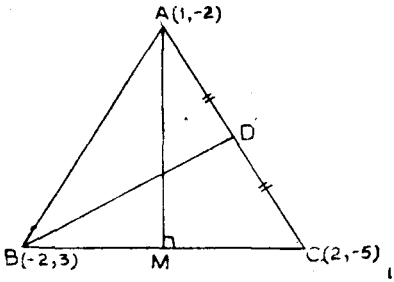


Fig. 8·12

and since AM is perpendicular to seg BC, slope of seg AM $= \frac{1}{2}$.

Since AM passes through the point A $(1,-2)$ and has slope $\frac{1}{2}$, its equation is $y-(-2)=\frac{1}{2}(x-1)$

$\therefore$ $2(y+2)=x-1$

$\therefore$ $x-2y-5=0$ is the equation of the altitude AM .... (i)

(ii) Since D is the midpoint of seg AC

$$\therefore \text{D} \equiv \left(\frac{2+1}{2}, \frac{-2-5}{2}\right) \equiv \left(\frac{3}{2}, -\frac{7}{2}\right)$$

$\therefore$ Median BD passes through the points

$$B\ (-2, 3) \text{ and } D\left(\frac{3}{2}, -\frac{7}{2}\right)$$

$\therefore$ Equation of line BD is

$$\frac{y-3}{x-(-2)} = \frac{3-\left(-\frac{7}{2}\right)}{-2-\frac{3}{2}}$$

$$\therefore \quad \frac{y-3}{x+2} = \frac{13}{-7} \quad \therefore \quad 13x+26 = -7y+21$$

$\therefore$ $13x+7y+5=0$ is the equation of median BD ....(ii)

Ex. 2. $A \equiv (-2, -4)$ and $B \equiv (6, 2)$ are two points Find the equation of the perpendicular bisector of seg AB.

Ans. Let M be the mid point of AB

$$\therefore \quad M \equiv \left(\frac{-2+6}{2}, \frac{-4+2}{2}\right) \equiv (2, -1).$$

Now the required line passes through M and is perpendicular to seg AB.

Since slope of seg $AB = \frac{2-(-4)}{6-(-2)} = \frac{6}{8} = \frac{3}{4}$

$\therefore$ Slope of the perpendicular bisector of seg AB $= -\frac{4}{3}$

$\therefore$ The perpendicular bisector of seg AB has slope $\left(-\frac{4}{3}\right)$ and it passes through the point (2, -1).

$\therefore$ Its equation is $y-(-1) = -\frac{4}{3}(x-2)$.

$\therefore$ $3(y+1) = -4(x-2)$.

$\therefore$ $4x+3y=5$ is the required equation.

Exercise 8.3 :

(1) Fill in the blanks in the following :

(a) Seg AB has slope 2 and A is (0, 1). $\therefore$ Equation of line AB is $y-$ —— $=$ ——$(x-$ ——$)$.

(b) Slope of seg PQ is $-1/3$. $\therefore$ The equation of the line through $(-2, -1)$ and perpendicular to seg PQ is $y+$ —— $=$ ——$(x+$ ——$)$.

(c) Slope of seg AB is 5. $\therefore$ Equation of the line through (0, 0) and parallel to seg AB is ——.

(2) A(2, -5), B(-2, 1) and C(4, 7) are the verices of the triangle ABC. Find the equation of (i) line BC; (ii) altitude from A; (iii) median from B; (iv) perpendicular bisecor of seg AB.

(3) A(1, 2), B(3, -4), C(-2, 5) and D(0, 1) are the vertices of the quadrilateral ABCD. Find the equation of (i) line AB, (ii) line CD; (iii) diagonal AC; (iv) diagonal BD

(4) Find the equation of the line passing through the mid point of seg AB and the midpoint of seg CD where $A \equiv (-1, -5)$; $B \equiv (3, 3)$; $C \equiv (2, -5)$; $D \equiv (-4, -1)$.

(5) $A \equiv (2, -4)$ and $B \equiv (-1, -7)$. Find the equation of
(i) the line through $(2, -5)$ and perpendicular to seg AB,
(ii) the line through the origin and parallel to seg AB.

(6) ABCD is a square and $A \equiv (2, -4)$ and $C \equiv (-4, -6)$. Find the equation of lines AC and BD. (Hint: seg BD is the perpendicular bisector of seg AC).

(7) $A \equiv (-3, -5)$ and $B \equiv (3, 4)$. Find (i) the equation of line AB, (ii) the equation of the line through B and perpendicular to line AB, (iii) the equation of the line through the origin and parallel to line AB.

(8) A(3, -4), B(-1, 6) and C(-5, -8) are the vertices of the triangle ABC. Find the equations of the lines joining the midpoints of the sides of the triangle.

§ 8·24. Standard forms of a line (contined):

Theorem 8·3: Slope-intercept form:

The equation of a line having slope m and making an intercept c on y axis is $y = mx + c$.

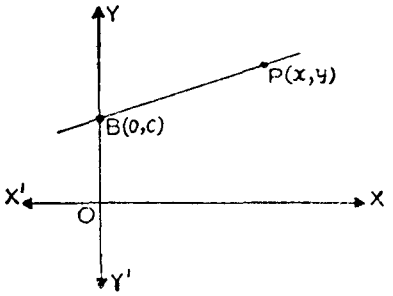


Fig. 8·13

Proof: Let P (x, y) be any point on the given line. Let it meet the y axis at B.

Since B lies on y axis, x coordinate of B is zero. Since intercept made by the line on y axis is c, y coordinate of B is c.

$\therefore$ B $\equiv (0, c)$.

$\therefore$ Slope of seg PB is

$$\frac{y-c}{x-0} = \frac{y-c}{x} \quad \ldots (i)$$

But slope of seg PB $= m$ $\quad \ldots (ii)$

From (i) and (ii), $\frac{y-c}{x} = m$. $\therefore y - c = mx$, $y = mx + c$

$\therefore$ $y = mx + c$ is the required equation.

Note : This standard form can be used only when the intercept given is y-intercept.

Theorem 8·4 : Two intercepts form :

The equation of a line making intercepts a and b (where $a \neq 0$ and $b \neq 0$) on the x and y axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$.

Proof: Let P (x, y) be any point on the given line. Let the line intersect the x axis at A and the y axis at B.

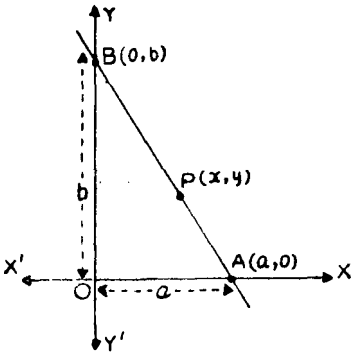


Fig. 8·14

Since x-intercept is a and y-intercept in b, A $\equiv (a, 0)$ and B $\equiv (0, b)$.

Since A, B, P are collinear,

$\therefore$ Slope of seg AP = slope of seg AB

$\therefore$ $\frac{y-0}{x-a} = \frac{b-0}{0-a}$

$\therefore$ $b(x-a) = -a(y)$

$\therefore$ $bx + ay = ab$.

Since $ab \neq 0$, dividing both sides by ab, $\frac{bx}{ab} + \frac{ay}{ab} = 1$

$\therefore$ $\frac{x}{a} + \frac{y}{b} = 1$ is the required equation.

§ 8·25. Examples :

Ex. 1. A line makes equal intercepts on the coordinate axes and passes through (2, 4). Find its equation.

Ans. Let the x-intercept of the line be ' a '

$\therefore$ Its y-intercept is also ' a '.

$\therefore$ Its equation is $\frac{x}{a} + \frac{y}{a} = 1$, i. e. $x + y = a$ .... (i)

Since (2, 4) lies on the line, substituting $x = 2$ and $y = 4$ in (i) we get $2 + 4 = a$ $\therefore$ $a = 6$.

$\therefore$ By (i), the equation of the line is $x + y = 6$.

Ex. 2. The x-intercept of a line is double its y-intercept. If it passes through (2, -4), find its equation.

Ans. Let the y-intercept of the line $= b$.

$\therefore$ Its x intercept $= 2b$.

$\therefore$ Its equation is $\frac{x}{2b}+\frac{y}{b}=1$, i. e. $x+2y=2b$ .... (i)

Since $(2,-4)$ lies on it, putting $x=2$ and $y=-4$ in (i), we get $2-8=2b$, $\therefore$ $b=-3$.

$\therefore$ By (i) the equation of the line is

$$x+2y=-6, \text{ i. e. } x+2y+6=0.$$

Ex. 3. A line passes through $(1, 3)$ and the sum of its intercepts on the axes is 8. Find its equation.

Ans. Let the intercepts made by the given line on the x and y axes be a and b respectively.

$\therefore$ $a+b=8$ .... (i)

Also the equation of the line is $\frac{x}{a}+\frac{y}{b}=1$ and since $(1, 3)$ lies on it, $\frac{1}{a}+\frac{3}{b}=1$. $\therefore$ $b+3a=ab$ .... (ii)

From (i) $b=8-a$. Putting this in (ii) we get

$\therefore$ $(8-a)+3a=a(8-a)$ $\therefore$ $8+2a=8a-a^2$

$\therefore$ $a^2-6a+8=0$ $\therefore$ $(a-2)(a-4)=0$ $\therefore$ $a=2$ or 4.

By (i) when $a=2$, $b=6$ and when $a=4$, $b=4$.

$\therefore$ The required line is either $\frac{x}{2}+\frac{y}{6}=1$ or $\frac{x}{4}+\frac{y}{4}=1$,

i. e. either $3x+y=6$ or $x+y=4$.

Exercise 8·4 :

(1) Fill in the blanks :

(a) A line has slope 3 and its y intercept is 2.
$\therefore$ Its equation is ———.

(b) A line makes intercept (-2) on the y-axis and is parallel to a line with slope -5. $\therefore$ Its equation is ———.

(c) A line makes intercept (-4) on y axis and is perpendicular to a line with slope $\frac{1}{2}$
$\therefore$ Its equation is ———.

(d) A line passes through $(2, 0)$ and $(0, -4)$.
$\therefore$ Its equation is ———.

(e) A line makes intercepts 2, 3 on the x and y axis respectively.
$\therefore$ Its equation is ———.

(2) A line makes intercepts of equal magnitude and opposite signs on the coordinate axes. If it passes through $(-4, 3)$, find its equation.

(3) The sum of the intercepts made by a line on the coordinate axes is 2. If it passes through (4, −3), find its equation.

(4) The x-intercept of a line exceeds its y-intercept by 3. If it passes through (10, −2), find its equation.

(5) A line intersects the coordinate axes at A and B such that the area of triangle OAB is 12 sq. units. If the line passes through (2, 3), find its equation.

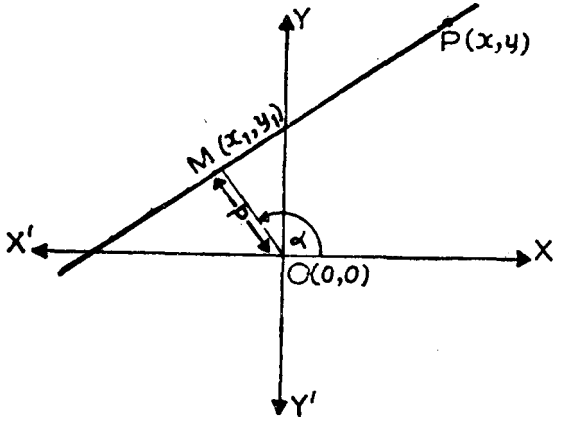
(6) A straight line cuts off positive intercepts from the coordinate axes such that the sum of the intercepts is 7. If the line passes through (−3, 8), find its equation.

(7) A line intersects the x axis at A and y axis at B. If the mid point of seg AB is (3, 4), find its equation.

§ 8·25. The normal form of a straight line :

Theorem 8·5 : If the perpendicular drawn from the origin to a line has length p and inclination α, then the equation of the line is $x\cos\alpha + y\sin\alpha = p$.

Proof :



Fig· 8·15

Let P (x, y) be any point on the given line. Draw OM perpendicular to the line meeting it at M. We are given that $l(\text{OM}) = p$ and $m\angle\text{XOM} = \alpha$.

Let M $\equiv (x_1, y_1)$. Since the angle α is in the standard position, by the definition of the trigonometric ratios,

$$\cos\alpha = \frac{x_1}{p} \text{ and } \sin\alpha = \frac{y_1}{p}.$$

Hence $x_1 = p\cos\alpha$, $y_1 = p\sin\alpha$ and we get

M $\equiv (p\cos\alpha, p\sin\alpha)$

$$\therefore \text{ Slope of PM} = \frac{y-y_1}{x-x_1} = \frac{y - p\sin\alpha}{x - p\cos\alpha} \quad \ldots\ldots \text{(i)}$$

Since the inclination of seg OM is α, slope of seg OM $= \tan\alpha$.

But seg PM is perpendicular to seg OM.

$$\therefore \text{ Slope of seg PM} = \frac{-1}{\tan\alpha} = -\cot\alpha \quad \ldots\ldots \text{(ii)}$$

By (i) and (ii), $\dfrac{y - p\sin\alpha}{x - p\cos\alpha} = -\cot\alpha.$

$$\therefore \frac{y - p\sin\alpha}{x - p\cos\alpha} = \frac{-\cos\alpha}{\sin\alpha}$$

$$\therefore y\sin\alpha - p\sin^2\alpha = -x\cos\alpha + p\cos^2\alpha.$$

$$\therefore x\cos\alpha + y\sin\alpha = p\sin^2\alpha + p\cos^2\alpha$$

$$= p(\sin^2\alpha + \cos^2\alpha).$$

$\therefore x\cos\alpha + y\sin\alpha = p$ is the required equation.

e. g. If the length of the perpendicular from the origin to the line is 5 and the inclination of the perpendicular is $\frac{\pi}{6}$, the equation of the line is $x\cos\frac{\pi}{6} + y\sin\frac{\pi}{6} = 5$

$$\therefore x\cdot\frac{\sqrt{3}}{2} + y\cdot\frac{1}{2} = 5$$

$$\therefore \sqrt{3}\,x + y = 10.$$

§ 8·3. General equation of a line:

We have proved five standard forms for the equation of a line which are as follows:

No.	name	data about the line	equation
1	slope-point form.	slope $= m$, one point (x_1, y_1)	$y - y_1 = m(x - x_1)$
2	two points form	two points (x_1, y_1) and (x_2, y_2)	$\dfrac{y-y_1}{x-x_1} = \dfrac{y_1-y_2}{x_1-x_2}$

No.	name	data about the line	equation
3	slope-intercept form.	slope $= m$, y – intercept $= c$	$y = mx + c$
4	two intercepts form.	x-intercept $= a$. y-intercept $= b$.	$\frac{x}{a} + \frac{y}{b} = 1$
5	normal form.	length of perpendicular from origin $= p$ and inclination of perpendicular $= \alpha$	$x \cos \alpha + y \sin \alpha = p$

We notice that each equation given above is a linear equation in x and y, i. e. the terms involving x and y are of first degree in x and y. Hence each one of the equations is of the form $Ax + By + C = 0$ where A, B, C are real numbers.

Hence we can say that the equation of a straight line is of the form $Ax + By + C = 0$, where A, B, C $\in$ R.

Now the converse of this is also true, i. e. $Ax + By + C = 0$ always represents a straight line.

We state this result as a theorem to be assumed without proof:

Theorem 8·6: $Ax + By + C = 0$ represents, in general, a straight line where A, B, C $\in$ R and A and B are not both zero.

§ 8·31. Slope and intercepts of $Ax + By + C = 0$:

According to theorem 8·6, $Ax + By + C = 0$ is a straight line when both A and B are not zero,

Now suppose $B = 0$. $\therefore$ $A \neq 0$ and the line beeomes

$$Ax + C = 0. \quad \text{i. e. } x = -\frac{C}{A}.$$

We know that this is a line parallel to y axis and has no slope.

Now if $B \neq 0$, we can write the equation $Ax + By + C = 0$ as

$$By = -Ax - C.$$

$$\therefore \quad y = \left(\frac{-A}{B}\right) x + \left(\frac{-C}{A}\right)$$

which is of the form $y = mx + C$ with $m = \frac{-A}{B}$. But in this form m means the slope of the line.

$\therefore$ $Ax+By+C=0$ is a straight line with slope $\frac{-A}{B}$ provided $B \neq 0$.

Now, to find the intercepts on the axes made by $Ax+By+C=0$, we proceed as follows :

To find the x-intercept, we must find the x coordinate of the point on the line where it meets the x axis. For this point the y coordinate $=0$. $\therefore$ Putting $y=0$ in $Ax+By+C=0$ we get :

$$x=\frac{-C}{A}, \text{ if } A \neq 0.$$

$\therefore$ x-intercept of the line $=\frac{-C}{A}$, if $A \neq 0$.

If $A=0$, the line is $By+C=0$ which is parallel to x axis.

Similarly, by putting $x=0$, we shall obtain the y-intercept of the line $=\frac{-C}{B}$, if $B \neq 0$.

If $B=0$, the line is $Ax+C=0$ which is parallel to y axis.

§ 8·32. Intersection of two lines :

If $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$ are two straigtht lines which are not parallel, then their point of intersection can be obtained by solving their equations simultaneously for x and y.

e. g. to find the point of intersection of

$$x+2y-5=0 \quad \ldots\ldots \text{(i)}$$

and $$3x+4y-11=0 \quad \ldots\ldots \text{(ii)}$$

we solve them as follows :

Multiplying (i) by 2 we get $2x+4y-10=0$ $\ldots\ldots$ (iii)

Substracting (iii) from (ii) we get $x-1=0$ $\therefore$ $x=1$.

From (i) $1+2y-5=0$ $\therefore$ $y=2$.

$\therefore$ The lines intersect at $(1, 2)$.

§ 8·33. Examples :

Ex. 1. Find the equation of a line through $(1, -3)$ and parallel to the line $3x+4y=1$.

Ans. The slope of $3x+4y=1$ is $-\frac{3}{4}$. $\therefore$ The line whose equation is required has slope $-\frac{3}{4}$ and it passes through the point $(1, -3)$.

$\therefore$ Its equation is $y+3=-\frac{3}{4}(x-1)$

$\therefore \quad 4y+12=-3x+3$

$\therefore \quad 3x+4y+9=0$ is the required equation.

Ex. 2. Find the equation of line a whose x-intercept is 4 and which is perpendicular to the line $x-2y+4=0$.

Ans. The slope of the line $x-2y+4=0$ is $-\frac{1}{(-2)}=\frac{1}{2}$.

$\therefore$ Slope of the required line is (-2) as it is perpendicular to $x-2y+4=0$.

Also the required line has x-intercept 4, i.e. it passes through $(4, 0)$.

$\therefore$ It equation is $y-0=-2(x-4) \quad \therefore \quad y=-2x+8$

$\therefore \quad 2x+y-8=0$ is the required equation,

Ex. 3. Find the equation of the line which passes through $A(1, 2)$ and also the mid point of the portion of the line $3x-4y+24=0$ intercepted between the axes of coordinates.

Ans. Let the line $3x-4y+24=0$ intersect the x axis at B and they y axis at C. Seg BC is the portion of the line intercepted between the axes.

To find the coordinates of B, put $y=0$ in the equation $3x-4y+24=0 \quad \therefore \quad 3x+24=0 \quad \therefore \quad x=-8 \quad \therefore \quad B \equiv (-8, 0)$.

Similarly to find the coordinates of C, we put $x=0$ in $3x-4y+24=0 \quad \therefore \quad -4y+24=0 \quad \therefore \quad y=6 \quad \therefore \quad C \equiv (0, 6)$.

The mid point of BC is $\left(\frac{-8+0}{2}, \frac{0+6}{2}\right)$, i. e. $(-4, 3)$.

Hence the required line passes through $(1, 2)$ and $(-4, 3)$.

$\therefore$ Its equation is $\quad \frac{y-2}{x-1}=\frac{2-3}{1-(-4)}$

$\therefore \quad \frac{y-2}{x-1}=\frac{-1}{5}$

$\therefore \quad 5y-10=-x+1$

$\therefore \quad x+5y-11=0$ is the required equation.

Ex. 4. Find the equation of a line which passes through the point of intersection of the lines $x+2y-3=0$ and $3x+4y-5=0$ and which is perpendicular to the line $x-3y+5=0$.

Ans. To find the point of intersection of

$x+2y-3=0 \quad \ldots\ldots \quad \ldots\ldots \quad (i)$

and $3x+4y-5=0 \quad \ldots\ldots \quad \ldots\ldots \quad (ii)$

Multiplying (i) by 2, we get

$$2x+4y-6=0 \quad \ldots\ldots \text{(iii)}$$

Subtracting (iii) from (ii) we get $x+1=0 \quad \therefore\ x=-1$

From (i) $-1+2y-3=0 \quad \therefore\ y=2.$

$\therefore$ The point of intersection of (i) and (ii) is $(-1, 2)$.

Also the slope of the line $x-3y+5=0$ is $-\frac{1}{(-3)}=\frac{1}{3}$.

Since the required line is perpendicular to this line, its slope $=-3$.

Hence the required line has slope (-3) and it passes through $(-1, 2)$.

$\therefore$ Its equation is $y-2=-3(x+1)$

$\therefore \quad y-2=-3x-3$

$\therefore \quad 3x+y+1=0$ is the required equation.

Ex. 5. If A (0, 0) ; B (4, 8) and C (3, 9) are the vertices of triangle ABC, find the equations of the altitudes BM and CN of the triangle and hence find the coordinates of its orthocentre.

Ans. (i) We shall find the equation of line BM.

Since slope of $AC=\frac{9-0}{3-0}=3$ and seg BM is perpendicular to seg AC, slope of seg $BM=-\frac{1}{3}$.

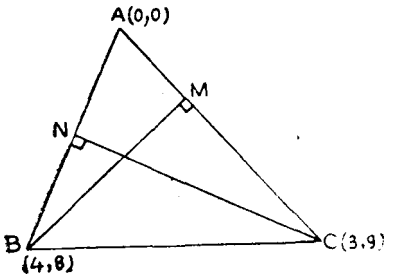


Fig. 8·16

Hence line BM has slope $-\frac{1}{3}$ and it passes through the point B (4, 8).

$\therefore$ Equation of line BM is

$\therefore \quad y-8=-\frac{1}{3}(x-4)$

$\therefore \quad 3y-24=-x+4$

$\therefore \quad x+3y-28=0$ is the equation of line BM. .... (i)

(ii) We now find the equation of CN.

Since slope of seg $AB=\frac{8-0}{4-0}=2$ and seg CN is perpendicular to seg AB, slope of seg $CN=-\frac{1}{2}$.

Hence line CN has slope $(-\frac{1}{2})$ and it passes through the point C (3, 9).

$\therefore$ Equation of line CN is $y-9=-\frac{1}{2}(x-3)$

$\therefore\ 2y-18=-x+3$

$\therefore\ x+2y-21=0$ is the equation of line CN. .... (ii)

(iii) To find the orthocentre of the triangle, we must find the point of intersection of seg BM and seg CN, i. e. we must solve the equations (i) and (ii).

Subtracting (ii) from (i) we get $y-7=0 \quad \therefore\ y=7.$

From (i) we get $x+21-28=0 \quad \therefore\ x=7.$

$\therefore$ Orthocentre of triangle ABC is (7, 7).

Exercise 8·5 :

(1) Find the equation of the line

(a) through (2, 3) and parallel to $3x-4y-7=0$.

(b) through (3, -5) and perpendicular to $x+2y=1$.

(c) having x intercept 4 and parallel to $3x+2y+1=0$.

(d) having y intercept -2 and perpendicular to $3x+y=7$.

(e) parallel to $4x+2y+9=0$ and passing through the point of intersection of the lines $3x-y-1=0$ and $5x+3y-11=0$.

(f) perpendicular to $8x-4y-9=0$ and passing through the point where $2x-y-7=0$ cuts the y axis.

(2) P is the point of intersection of the lines $7x-y-32=0$ and $3y-2x+1=0$. Lines are drawn through P making intercepts of *equal magnitude* on the coordinate axes. Find the equations of these lines.

(3) Find the coordinates of the foot of the perpendicular drawn from (1, 2) to the straight line $x-3y+7=0$.

(4) Find the equation of the line passing through the point (3, 5) and

(a) passing through the point of intersection of the lines $x+y=12$ and $y-x=2$,

(b) passing through the point which bisects the portion of the line $3x+4y=24$ intercepted between the coordinates axes,

(c) passing through the point which divides internally in the ratio 1 : 3, the portion of the lines $x+4y=24$ intercepted between the coordinate axes.

(5) Find the coordinates of the centroid and the orthocentre of the triangle formed by the lines $3x-4y=17$, $y=4$ and $12x+5y=12$.

(6) A (2, 1), B (5, 3) and C $(-1, 3)$ are the vertices of the triangle ABC. Find (i) the equation of the median AD, (ii) the equation

of the altitude BE and (iii) the coordinates of the point common to seg AD and seg BE.

(7) A is the vertex and seg BC is the base of the isosceles triangle ABC. If the coordinates of A, B and C are respectively $(2, k)$, $(5, 4)$ and $(1, 0)$, find the value of k. Also find the equation of the median AD of the triangle.

(8) Show that the triangle whose sides are $x=2$, $y=3$ and $4x+3y=29$ is a right-angled triangle. Hence, find the coordinates of its circumcentre and orthocentre.

(9) Find the equations of the altitudes of the triangle formed by the lines $4x-y=4$, $2x-y=6$ and $x+y=6$ and show that the orthocentre of the triangle is $(8, 1)$.

(10) Show that $(5, 5)$ is the orthocentre of the triangle formed by the lines $x+y=9$, $2x-y=9$ and $2y-x=9$.

(11) The coordinates of the vertex A of a square ABCD are $(1, 2)$ and the equation of the diagonal BD is $x+2y=10$. Find the equation of the other diagonal and the coordinates of the centre of the square.

(12) The ends of the base of an isosceles triangle are at $(4, 0)$ and $(0, 2)$, the equation of one of the equal sides is $x=4$. Find the coordinates of the third vertex of the triangle.

(13) B $(4, 0)$ and C $(6, 4)$ are the extremities of the base of an isosceles triangle ABC. If the equation of one of the sides is $x-y=4$, find the equation of the other side.

(14) A straight line $4x+3y=24$ cuts the x axis in L and the y axis in M. The rectangle OLNM is completed (O is the origin). Find the equation of line ON and the coordinates of the centre of this rectangle.

(15) Two vertices of a rectangle are $(-2, 1)$ and $(2, 4)$. Equation of one side is $3x-4y-10=0$, find (i) the area of the rectangle, (ii) equation of the diagonal through the vertex $(2, 4)$, (iii) coordinates of the point where the diagonals meet.

(16) Find p, if the line joining $(4, 5)$ and $(p, -5)$ is bisected by the line $2x+3y-6=0$.

Angle between two lines :

Theorem 8·7 : If m_1 and m_2 are the slopes of two lines such that $m_1 \cdot m_2 \neq -1$, then the measure of the acute angle θ between the lines is given by $\tan\theta = \left|\dfrac{m_1-m_2}{1+m_1 m_2}\right|$

Proof : Let the inclination of the line with slope m_1 be θ_1 and the inclination of the line with slope m_2 be θ_2.

$\therefore \tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.

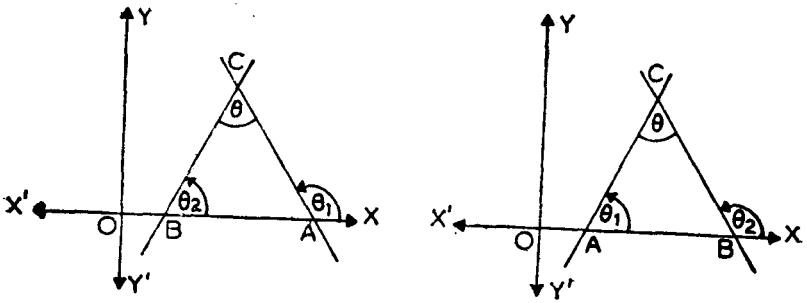


Fig. 8·17

Let the line with slope m_1 intersect the x axis at A and the line with slope m_2 intersect the x axis at B. Let the lines intersect each other at C.

Let $m\angle ACB = \theta$ then from fig. 4·17, we have,

either $\theta = \theta_1 - \theta_2$ or $\theta = \theta_2 - \theta_1$

$\therefore \tan\theta = \tan(\theta_1 - \theta_2)$ or $\tan\theta = \tan(\theta_2 - \theta_1)$

$= -\tan(\theta_1 - \theta_2)$

$\therefore \tan\theta = |\tan(\theta_1 - \theta_2)|$

$\therefore \tan\theta = \left|\dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}\right|$

$\therefore \tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$ as required.

Note: (i) If $m_1 m_2 = -1$, the lines are perpendicular and $m\angle\theta = 90°$ or $\pi/2$, as proved in § 8·13.

(ii) If $m_1 = m_2$, the lines are parallel and $\tan\theta = 0$, $\therefore m\angle\theta = 0$.

Corollary: If θ is the acute angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ where $b_1 \neq 0$, $b_2 \neq 0$ and $a_1a_2 + b_1b_2 \neq 0$, then $\tan\theta = \left|\dfrac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}\right|$

Proof: Since the slopes of the lines are $\dfrac{-a_1}{b_1}$ and $\dfrac{-a_2}{b_2}$, from theorem 8·7, we have

$$\tan\theta = \left|\frac{\left(\frac{-a_2}{b_2}\right) - \left(\frac{-a_1}{b_1}\right)}{1 + \left(\frac{-a_2}{b_2}\right)\cdot\left(\frac{-a_1}{b_1}\right)}\right|$$

$$= \left|\frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}\right| \text{ as required.}$$

§ 8·41. Examples:

Ex. 1. Find the acute angle between the lines $6x-2y-7=0$ and $4x+2y+1=0$.

Ans. The slopes af the lines are given by

$$m_1=-\frac{6}{(-2)}=3 \text{ and } m_2=-\frac{4}{2}=-2.$$

If θ is the acute angle between them,

$$\tan\theta=\left|\frac{3-(-2)}{1+3(-2)}\right|=\left|\frac{5}{-5}\right|=1$$

$$\therefore\ \theta=\frac{\pi}{4}.$$

Ex. 2. Find the equation of a line through (1, 4), which makes an angle of 45° with the line $2x+y-5=0$.

Ans. Let m be the slope of the line whose equation is to be found.

Since the slope of $2x+y-5=0$ is $-\frac{2}{1}=-2$ and the measure of the angle between the two lines is 45°, we have

$$\tan 45°=\left|\frac{m-(-2)}{1+m(-2)}\right|$$

$$\therefore\ 1=\left|\frac{m+2}{1-2m}\right| \quad\therefore\quad \frac{m+2}{1-2m}=\pm 1 \quad \text{(Note this step)}$$

Case (i): If $\frac{m+2}{1-2m}=1$ then $m+2=1-2m$

$$\therefore\ 3m=-1 \quad\therefore\quad m=-\frac{1}{3}$$

Case (ii): If $\frac{m+2}{1-2m}=-1$ then $m+2=-1+2m$

$$\therefore\ m=3$$

Hence the required line passes through (1, 4) and its slope is either 3 or $-\frac{1}{3}$.

$\therefore$ Its equation is $y-4=3(x-1)$ or $y-4=-\frac{1}{3}(x-1)$

i. e. $y-4=3x-3$ or $3y-12=-x+1$

i. e. $3x-y+1=0$ or $x+3y-13=0$

is the equation of the required line.

Ex. 3. $(1,-4)$ and $(-2,8)$ are the coordinates of the opposite vertices of a square. Find the equations of its sides.

Ans. Let ABCD be the square where $A \equiv (1,-4)$; $C \equiv (-2,8)$

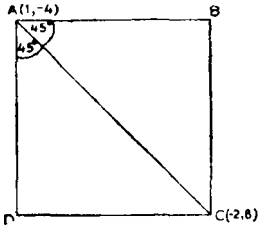


Fig. 8·18

$\therefore$ Slope of seg AC

$$= \frac{8-(-4)}{-2-1} = -4.$$

We know that $m\angle BAC = 45°$ and $m\angle DAC = 45°$.

Let the slope of seg AB (or AD) be m. Since

$m\angle BAC$ (or $m\angle DAC$) $= 45°$

$$\therefore \quad \tan 45° = \left|\frac{m-(-4)}{1+m(-4)}\right|$$

$$\therefore \quad 1 = \left|\frac{m+4}{1-4m}\right| \qquad \therefore \quad \frac{m+4}{1-4m} = \pm 1.$$

$\therefore \quad m+4 = \pm(1-4m)$

$\therefore$ Either $m+4 = 1-4m$ or $m+4 = -1+4m$

$\therefore$ Either $m = -\frac{3}{5}$ or $m = \frac{5}{3}$

These two values of m are the slopes of seg AB and seg AD.

Suppose slope of seg AB $= -\frac{3}{5}$ and slope of seg AD $= \frac{5}{3}$.

Since seg DC is parallel to seg AB, slope of seg DC $= -\frac{3}{5}$.

Since seg BC is parallel to seg AD, slope of seg BC $= \frac{5}{3}$.

Hence the equations of the sides are found as follows:

(i) Line AB has slope $-\frac{3}{5}$ and it passes through the point $A(1,-4)$. $\therefore$ Its equation is $y+4 = -\frac{3}{5}(x-1)$

$$\therefore \quad 3x+5y+17 = 0 \qquad \ldots\ (i)$$

(ii) Line BC has slope $\frac{5}{3}$ and it passes through the point $C(-2,8)$. $\therefore$ Its equation is $y-8 = \frac{5}{3}(x+2)$,

$$\therefore \quad 5x-3y+34 = 0 \qquad \ldots\ (ii)$$

(iii) Line CD has slope $-\frac{3}{5}$ and it passes through the point C $(-2, 8)$. $\therefore$ Its equation is $y-8=-\frac{3}{5}(x+2)$,

$\therefore \quad 3x+5y-34=0 \quad \ldots\ldots$ (iii)

(iv) Line DA has slope $\frac{5}{3}$ and it passes through the point A $(1, -4)$. $\therefore$ Its equation is $y+4=\frac{5}{3}(x-1)$

$\therefore \quad 5x-3y-17=0 \quad \ldots\ldots$ (iv)

(i), (ii), (iii) and (iv) give the required answers.

Exercise 8·6 :

(1) Find the angle between the lines :

(a) $3x-y+4=0$ and $6x+3y-5=0$

(b) $x+3y+9=0$ and $2x+y+1=0$

(c) $4x+y+2=0$ and $x+y+3=0$

(d) $8x+2y+3=0$ and $2x-2y-5=0$

(e) $3x+y+7=0$ and $x-3y+7=0$

(f) $2x+y+4=0$ and $4x+2y+7=0$

(g) $\sqrt{3}x-y-4=0$ and $x-\sqrt{3}y+7=0$

(h) $x+\sqrt{3}y+12=0$ and $x-\sqrt{3}y-5=0$

(2) Find the equation of the lines through $(3, -5)$ making angles of measure 45° with the line $x+2y+1=0$.

(3) ABC is an equilateral triangle. If A $\equiv (1, 2)$ and the equation of BC is $x+y+5=0$, find the equations of AB and AC.

(4) If $(3, -5)$ and $(-1, 3)$ are the coordinates of the opposite vertices of a square, find the equations of its sides and diagonals.

(5) If $(6, 4)$ and $(0, 8)$ are the opposite vertices of a square, find the coordinates of the other vertices and the equations of its sides.

§8·5. Parametric equations of a straight line :

Consider a line passing through the point A (x_1, y_1) and having inclination θ

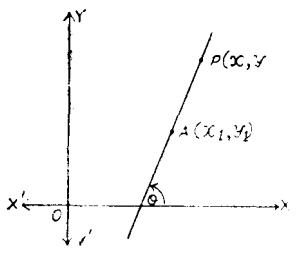


Fig. 8·19

Then its equation is $y-y_1=\tan\theta\cdot(x-x_1)$ where P (x, y) is any point on it. This equation can be written as :

$$y-y_1=\frac{\sin\theta}{\cos\theta}\cdot(x-x_1)$$

$$\therefore \quad \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \text{ say}$$

Since $\frac{x-x_1}{\cos\theta}=r$,

$\therefore \quad x=x_1+r\cos\theta.$

Since $\frac{y-y_1}{\sin\theta}=r$, $\quad \therefore \quad y=y_1+r\sin\theta.$

The equations $x=x_1+r\cos\theta$ and $y=y_1+r\sin\theta$ which give the coordinates of any point (x, y) on the line in terms of r, which is a variable, are called the parametric equations of the line. The variable 'r' is called the parameter.

Let us now find out what 'r' represents.

Since $x-x_1=r\cos\theta$ and $y-y_1=r\sin\theta$, we have

$$(x-x_1)^2+(y-y_1)^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2(\cos^2\theta+\sin^2\theta)=r^2.$$

$$\therefore r=\sqrt{(x-x_1)^2+(y-y_1)^2}.$$

But the R. H. S. represents the distance between the points (x, y) and (x_1, y_1). Hence r represents the distance of the variable point P (x, y) from the point A (x_1, y_1).

This gives us the following important result : If a line passes through A (x_1, y_1) and has inclination θ then the coordinates of any point P on the line are given by $(x_1+r\cos\theta,\ y_1+r\sin\theta)$ where $l(AP)=r$. Similarly, if Q is $(x_1-r\cos\theta, y_1-r\sin\theta)$, then $l(AQ)=r$. This happens because a point on a line at distance r from A may lie on either side of A.

Hence if $l(AP)=r$, P has two positions given by $(x_1\pm r\cos\theta, y_1\pm r\sin\theta)$.

e. g. If a line passes through A (1, 5) and has inclination $\frac{\pi}{6}$ then the two points on the line at a distance of 4 units from A are given by

$$\left(1\pm 4\cos\frac{\pi}{6}, 5\pm 4\sin\frac{\pi}{6}\right)\equiv\left(1\pm 4\cdot\frac{\sqrt{3}}{2}, 5\pm 4\cdot\frac{1}{2}\right)$$

$$\equiv(1\pm 2\sqrt{3}, 5\pm 2)$$

Hence the required points are $(1+2\sqrt{3}, 7)$ and $(1-2\sqrt{3}, 3)$

§ 8·51. Perpendicular distance from a point to a line :

Theorem 8·8 : If P (x_1, y_1) is any point and $ax+by+c=0$ is a line then the perpendicular distance of P from the line is given by : $\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Proof: Let $P(x_1, y_1)$ be the given point and $ax+by+c=0$ be the given line. Draw PM perpendicular to the line meeting it at M. We have to find $l(PM)$.

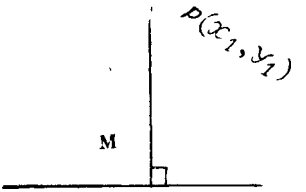


Fig 8·20

Let $l(PM)=r$.

If the inclination of seg PM is θ then
$M \equiv (x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

But M lies on the line $ax+by+c=0$

$\therefore a(x_1 \pm r\cos\theta)+b(y_1 \pm r\sin\theta)+c=0$

$\therefore ax_1 \pm ar\cos\theta+by_1 \pm br\sin\theta+c=0$

$\therefore \pm r(a\cos\theta+\sin\theta)+ax_1+by_1+c=0$

$\therefore ax_1+by_1+c = \pm r(a\cos\theta+b\sin\theta)$

$$\therefore r = \pm \frac{ax_1+by_1+c}{a\cos\theta+b\sin\theta}$$

$$\therefore |r| = \left|\frac{ax_1+by_1+c}{a\cos\theta+b\sin\theta}\right| \quad \ldots\ (i)$$

Now the slope of $ax+by+c=0$ is $\frac{-a}{b}$ and since PM is perpendicular to it, slope of $PM = \frac{b}{a}$. But slope of $PM = \tan\theta$ since its inclination is θ.

$$\therefore \tan\theta = \frac{b}{a} \quad \therefore \sin\theta = \frac{b}{\sqrt{a^2+b^2}} \text{ and } \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

Substituting these values in (i), we get

$$|r| = \left|\frac{ax_1+by_1+c}{a\cdot\frac{a}{\sqrt{a^2+b^2}}+b\cdot\frac{b}{\sqrt{a^2+b^2}}}\right| = \left|\frac{ax_1+by_1+c}{\left(\frac{a^2+b^2}{\sqrt{a^2+b^2}}\right)}\right|$$

$$\therefore |r| = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right| \text{ as required.}$$

§ 8·52. Examples:

Ex. 1. Find the distance of the point $(2,-1)$ from the line $3x+4y-3=0$.

Ans. Required distance $= \left|\frac{3(2)+4(-1)-3}{\sqrt{3^2+4^2}}\right|$

$$= \left|\frac{6-4-3}{5}\right| = \frac{1}{5}.$$

Ex. 2. Find the perpendicular distance between the parallel lines $3x-4y-16=0$ and $3x-4y-6=0$.

Ans. The perpendicular distance between two parallel lines is the distance of any point on one of them from the other line.

Let us find a point on $3x-4y-6=0$.

Suppose we put $y=0$. $\therefore 3x-6=0$ $\therefore x=2$

$\therefore$ $(2, 0)$ lies on $3x-4y-6=0$. Hence required distance is the distance of $(2, 0)$ from $3x-4y-16=0$.

$\therefore$ Required distance $= \left|\frac{3(2)-4(0)-16}{\sqrt{3^2+(-4)^2}}\right|$

$$= \left|\frac{6-16}{5}\right| = 2.$$

Ex. 3. Find the coordinates of a point on the line $x+y-5=0$ whose distance from the line $6x+8y+1=0$ is 3·5 units.

Ans. Let the required point on the line $x+y-5=0$ be $P(h, k)$.

$\therefore h+k-5=0$ .... (i)

Since the distance of $P(h, k)$ from $6x+8y+1=0$ is 3·5, we have $\left|\frac{6h+8k+1}{\sqrt{6^2+8^2}}\right| = 3{\cdot}5$

$\therefore \frac{6h+8k+1}{10} = \pm 3{\cdot}5$ $\therefore 6h+8k+1 = \pm 35$

$\therefore 6h+8k+1 \mp 35 = 0$

$\therefore 6h+8k+36=0$, i.e. $3h+4k+18=0$ .... (ii)

or $6h+8k-34=0$, i.e. $3h+4k-17=0$ .... (iii)

Solving (i) and (ii) we get $h=28$ and $k=-23$

$\therefore P \equiv (28, -23)$

Solving (i) and (iii) we get $h=3$ and $k=2$ $\therefore P \equiv (3, 2)$.

Hence the required point is $(28, -23)$ or $(3, 2)$.

Ex. 4. Find the equation of a line parallel to $x+2y-1=0$ and which is at a distance of $2\sqrt{5}$ units from the point $(1, 3)$.

Ans. The slope of the line $x+2y-1=0$ is $-\frac{1}{2}$ and since the required line is parallel to it, it also has slope $-\frac{1}{2}$.

Let the y-intercept of the required line be c.

$\therefore$ The equation of the required line will be $y=(-\frac{1}{2})x+c$

$\therefore\ x+2y-2c=0$ .... (i)

Since the distanee of $(1, 3)$ from this line is $2\sqrt{5}$, we have

$$\left|\frac{1+2(3)-2c}{\sqrt{1^2+2^2}}\right|=2\sqrt{5}$$

$\therefore\ \left|\frac{7-2c}{\sqrt{5}}\right|=2\sqrt{5}$ $\therefore\ \frac{7-2c}{\sqrt{5}}=\pm 2\sqrt{5}$

$\therefore\ 7-2c=\pm 10$ $\therefore\ 2c=7\pm 10$ $\therefore\ 2c=17$ or -3.

$\therefore$ From (i), the equation of the required line is

$$x+2y-17=0 \text{ or } x+2y+3=0.$$

Ex. 5. M $(2, 3)$ is the centre of a square whose side is of length 6 units. If one of the sides of this square is parallel to $3x+4y+1=0$, find the equations of the sides of the square.

Ans. Let ABCD be the square. Let lines AC and BD meet at M $(2, 3)$, the centre of the square.

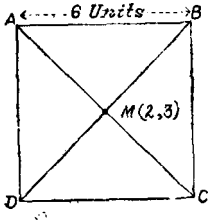


Fig. 8·21

Suppose line BC (and hence line AD) is parallel to $3x+4y+1=0$.

Since slope of this line is $-\frac{3}{4}$, slope of seg BC (and hence of seg AD) is also $-\frac{3}{4}$.

Let the y-intercept of line BC (or line AD) be c.

$\therefore$ Its equation is $y=\left(-\frac{3}{4}\right)x+c$. i. e. $3x+4y-4c=0$.

Since $l(AB)=6$ units, the perpendicular distance of M from seg BC (or seg AD) is 3 units.

Hence $\left|\frac{3(2)+4(3)-4c}{\sqrt{3^2+4^2}}\right|=3.$

$\frac{18-4c}{5}=\pm 3$ $\therefore\ 18-4c=\pm 15$

$\therefore\ 4c=18\pm 15$ $\therefore\ 4c=33$ or 3.

$\therefore \quad c = \frac{33}{4}$ or $\frac{3}{4}$

Hence one value of c is for seg BC and the other is for seg AD.

$\therefore$ The equation of line BC is $3x + 4y - 33 = 0$ and the equation of line AD is $3x + 4y - 3 = 0$.

Since line AB (or line CD) is perpendicular to line BC and slope of seg BC is $-\frac{3}{4}$, slope of seg AB (or seg CD) is $\frac{4}{3}$. If the y intercept of line AB (or line CD) is c', then the equation of line AB (or line CD) is $y = \frac{4}{3}x + c'$, i. e. $4x - 3y + 3c' = 0$.

Again the perpendicular distance of M (2, 3) from this line is 3 units.

$$\therefore \quad \left| \frac{4(2) - 3(3) + 3c'}{\sqrt{4^2 + 3^2}} \right| = 3$$

$$\therefore \quad \frac{3c' - 1}{5} = \pm 3$$

$$\therefore \quad 3c' - 1 = \pm 15 \qquad \therefore \quad 3c' = 1 \pm 15$$

$$\therefore \quad 3c' = 16 \text{ or } -14 \qquad \therefore \quad c' = \frac{16}{3} \text{ or } -\frac{14}{3}.$$

As before, one value of c' is for line AB and the other is for line CD.

$\therefore$ The equation of line AB is $4x - 3y + 16 = 0$ and the equation of line CD is $4x - 3y - 14 = 0$.

Hence the sides of the square have the equations :

$3x + 4y - 33 = 0$; $3x + 4y - 3 = 0$; $4x - 3y + 16 = 0$ and $4x - 3y - 14 = 0$.

Exercise 8.7 :

(1) Find the distance of $(1, -2)$ from the line $12x + 5y = 1$.

(2) Find the distance of $(-3, 7)$ from the line $3x - y = 4$.

(3) Find the perpendicular distance between the following pairs of parallel lines :

(i) $2x + y - 5 = 0$; $2x + y + 15 = 0$

(ii) $3x + 4y - 6 = 0$; $6x + 8y - 17 = 0$

(iii) $x + y + 4 = 0$; $x + y - 6 = 0$

(4) Find the coordinates of a point on $x+y+3=0$ whose distance from $x+2y+2=0$ is $\sqrt{5}$ units.

(5) Find the coordinates of a point on $2x+y-3=0$ whose distance from $6x-8y-3=0$ is 0·5 units.

(6) Find the coordinates of a point on the x axis whose distance from $5x+12y-12=0$ is one unit.

(7) Find the equation of a line parallel to $x+3y-5=0$ and which is at a distance of $2\sqrt{10}$ units from the point $(1, -1)$.

(8) Find the equation of a line parallel to $3x-4y-7=0$ and which is at a distance of 4 units from the point $(-2, 1)$.

(9) Since any point on the bisector of an angle is equidistant from the two arms of the angle, find the equations of the bisectors of the angles between $x+2y-1=0$ and $4x+2y-3=0$,

(10) Find the equations of the bisectors of the angles between the lines $3x-4y-6=0$ and $6x+8y+1=0$.

§ 8·6. Sign of the perpendicular :

A line divides the plane into two parts. If the equation of a line and the coordinates of any two points in the plane are given, we wish to find out whether the points lie on the same side of the line or on the opposite sides. To do this, we shall use the following convention :

"The perpendicular from the origin to a line is always taken to be positive provided the line does not pass through the origin."

Now the length of the perpendicular from the origin to a line $ax+by+c=0$, will be $\left|\frac{c}{\sqrt{a^2+b^2}}\right|$. Since $\sqrt{a^2+b^2}$ is to be taken positive, we may remove the modulus if c is positive. Hence to find the length of the perpendicular from origin to a line, we write the equation of the line so that c is positive and then $\frac{c}{\sqrt{a^2+b^2}}$ is the length of this perpendicular, e. g. if we want the length of the perpendicular from the origin to the line $3x+4y-10=0$, we rewrite the equation as $-3x-4y+10=0$ and then the length of the perpendicular $=\frac{10}{\sqrt{(-3)^2+(-4)^2}}=2$.

Now consider a line $ax+by+c=0$. If we write the equation with c as positive, then the perpendicular from the origin will be positive.

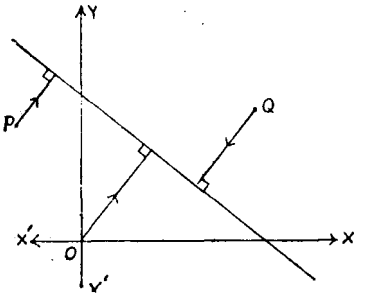


Fig. 8·22

Since the line devides the plane in two parts, these can be distinguished as : either the part containing the origin or the part not containing the origin. Now since the perpendicular from the origin is positive, if a point P is taken on the side of the line containing origin, the perpendicular from this point must be positive since the direction of this perpendicular is the same as that of the perpendicular from origin. Similarly, if Q is any point on the other side of the line, the perpendicular from Q to the line must be negative since its direction is opposite to that of the perpendicular from the origin. This means that if two points P and Q are taken on opposite sides of a line, the lengths of the perpendiculars must have opposite signs. We shall prove this as follows :

Theorem 8·9 : The line $ax+by+c=0$ divides the join of **A** (x_1, y_1) and B (x_2, y_2) in the ratio

$$-\left[\frac{ax_1+by_1+c}{ax_2+by_2+c}\right]$$ provided neither A nor B lies on the line.

Proof : Let the line $ax+by+c=0$ intersect seg AB at R.

Let $l(AR) : l(RB) = m : n$. $\therefore$ By section formula we have

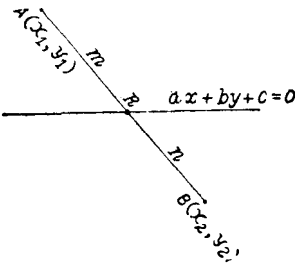


Fig. 8·23

$$R \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

But R lies on $ax+by+c=0$

$$\therefore a\cdot\frac{mx_2+nx_1}{m+n} + b\cdot\frac{my_2+ny_1}{m+n} + c = 0$$

$$\therefore amx_2+anx_1+bmy_2+bny_1+cm+cn=0$$

$$\therefore m(ax_2+by_2+c)+n(ax_1+by_1+c)=0$$

$$\frac{m}{n} = -\frac{ax_1+by_1+c}{ax_2+by_2+c}$$ as required.

Corollary : The points A (x_1, y_1) and B (x_2, y_2) lie on the

same or opposite side of the line $ax+by+c=0$, according as whether (ax_1+by_1+c) and (ax_2+by_2+c) are of same or opposite signs.

Proof: Let the line $ax+by+c=0$ divide seg AB in the ratio $m:n$.

$$\therefore \frac{m}{n}=-\frac{ax_1+by_1+c}{ax_2+by_2+c}.$$

Now if (ax_1+by_1+c) and (ax_2+cy_2+c) have the *same sign*, then $\frac{m}{n}$ is negative and hence $ax+by+c=0$ divides seg AB externally, i. e. A and B are on the *same side* of the line.

If (ax_1+by_1+c) and (ax_2+by_2+c) have *opposite signs*, then $\frac{m}{n}$ is positive and hence $ax+by+c=0$ divides seg AB internally, i. e. A and B are on *opposite sides* of the line as required.

Note: Now if $ax+by+c=0$ is written with c positive, then the perpendicular from (0, 0) on the line, viz. $\frac{c}{\sqrt{a^2+b^2}}$, is positive. Hence, if (ax_1+by_1+c) is positive, (x_1, y_1) lies on the origin side of the line and if (ax_1+by_1+c) is negative then (x_1, y_1) lies on the non-origin side of the line.

§ 8·61. **Examples:**

Ex. 1. Find whether (2, 5) lies on the origin side or non-origin side of the line $2x+3y-4=0$,

Ans. Rewrite the equation as $-2x-3y+4=0$.

Putting (2, 5) in $-2x-3y+4$ we get

$-2(2)-3(5)+4=-15$ which is nenative.

$\therefore$ The point (2, 5) lies on the non-origin side of this line.

Ex. 2. For what values of k will the point $(2, k)$ lie on the origin side of the line $3x-2y+4=0$?

Ans. Substituting $(2, k)$ in $3x-2y+4$ we get

$$3(2)-2(k)+4=10-2k.$$

The point $(2, k)$ lies on the origin side of the line if $(10-2k) > 0$, i. e., if $2k < 10$, i. e., if $k < 5$. Hence the point $(2, k)$ will on the origin side of the line $3x-2y+4=0$ if $k < 5$.

Excrcise 8·8 :

(1) Find whether (2, −1) lies on the origin or the non-origin side of $3x-y=7$.

(2) Find whether (5, 4) lies on the origin or the non-origin side of $x+y=5$.

(3) Find the set of values of k for which the point $(3, k)$ will lie on the origin side of $2x+y-1=0$.

(4) Find the set of values of k for which the point $(k, 5)$ will lie on the non-origin side of $x+3y-5=0$.

(5) Find the set of values of k for which (2, 3) will lie on the origin side of $x+ky+4=0$.

§ 8·62. Area of a triangle :

We shall now find the area of a triangle when either the coordinates of its vertices are given or the equations of its sides are given.

Suppose ABC is a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. To find the area of triangle ABC, we proceed as follows : Draw segs. AA′, BB′ and CC′ perpendicular to the x axis meeting it at A′, B′, C′. Then the figures AA′C′C, AA′B′B and CC′B′B are trapeziums.

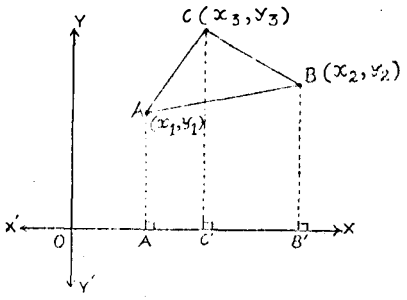


Fig. 8·24

∴ Area of triangle ABC = area of trapezium AA′C′C + area of trapezium CC′B′B − area of trapezium AA′B′B.

Now area of trapezium AA′C′C

$$= \tfrac{1}{2}\,[\,l(AA') + l(CC')\,] \times l(A'C')$$

$$= \tfrac{1}{2}\,(y_1+y_3)\,(x_3-x_1).$$

[Since $l(A'C') = l(OC') - l(OA') = x_3 - x_1$]

Similarly area of trapezium $CC'B'B = \tfrac{1}{2}(y_3+y_2)(x_2-x_3)$ and area of trapezium $AA'B'B = \tfrac{1}{2}(y_1+y_2)(x_2-x_1)$

∴ Area of triangle ABC

$$= \tfrac{1}{2}(y_1+y_3)(x_3-x_1) + \tfrac{1}{2}(y_3+y_2)(x_2-x_3) - \tfrac{1}{2}(y_1+y_2)(x_2-x_1)$$

$$= \tfrac{1}{2}\,[\,x_3(y_1+y_3)-x_1(y_1+y_3)+x_2(y_3+y_2)$$
$$-x_3(y_3+y_2)-x_2(y_1+y_2)+x_1(y_1+y_2)]$$
$$= \tfrac{1}{2}\,[\,x_1(y_1+y_2-y_1-y_3)+x_2(y_3+y_2-y_1-y_2)$$
$$+x_3(y_1+y_3-y_3-y_2\,]$$
$$= \tfrac{1}{2}\,[\,x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\,]$$ is the required area .... (i)

This may be remembered as a formula. In the examples that follow, we shall see another method for finding the area of a triangle.

§ 8·63. Directed area :

In the fig. 8·24 the points A, B, C were in anticlockwise direction. By convention, the area of a triangle will be positive if the points are in anticlockwise direction and negative if they are in clockwise direction. Thus, the area of triangle BAC in fig. 8·24 will have the same magnitude as the area of triangle ABC but will have a negative sign.

Since in an example, we shall be interested in finding the magnitude of the area, we shall take the absolute value of the R. H. S. of (i) while using it in the example.

§ 8·64. Examples :

Ex. 1. Find the area of triangle whose vertices are A(1, 2); B(−3, 4) and C(4, 6).

Ans. Let $(x_1, y_1) \equiv (1, 2)$; $(x_2, y_2) \equiv (-3, 4)$ and $(x_3, y_3) \equiv (4, 6)$.

Then area of triangle ABC

$$= \tfrac{1}{2}\,|\,x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\,|$$
$$= \tfrac{1}{2}\,|\,1(4-6)+(-3)(6-2)+4[2-4]\,|$$
$$= \tfrac{1}{2}\,|(-2)+(-12)+(-8)| = \tfrac{1}{2}\cdot 22 = 11 \text{ sq. unit}$$

Ex. 2. If A(3, 3); B(−5, 7) and C(−1, 4) are the vertices of a triangle, find the length of the perpendicular from A to BC and hence find the area of triangle ABC.

Ans. Here the equation of line BC is

$$\frac{y-7}{x+5} = \frac{7-4}{-5+1} \qquad \therefore \quad \frac{y-7}{x+5} = \frac{3}{-4}$$

$$\therefore\ 3x+15 = -4y+28 \qquad \therefore\ 3x+4y-13 = 0.$$

Hence if seg AM is the perpendicular to seg BC, we have

$$l(\mathrm{AM}) = \left| \frac{3(3)+4(3)-13}{\sqrt{3^2+4^2}} \right| = \frac{8}{5}$$

Also $l(\mathrm{BC}) = \sqrt{(-5+1)^2+(7-4)^2} = \sqrt{16+9} = 5$

$\therefore$ Area of triangle ABC $= \frac{1}{2}\, l(\mathrm{AM}) \cdot l(\mathrm{BC})$

$$= \frac{1}{2} \cdot \frac{8}{5} \cdot 5 = 4 \text{ sq. units.}$$

Exercise 8·9 :

(1) Find the area of the triangle whose vertices are (3, 1), (4, 2) and (5, 3).

(2) Find the area of the triangle whose vertices are $(-2, 1)$; $(-3, -5)$ and (2, 4).

(3) Find the area of the triangle whose vertices are the points $(-1, 2)$, (2, 4) and the origin.

(4) If ABC is a triangle where $A \equiv (3, -5)$; $B \equiv (-2, 1)$ and $C \equiv (1, 4)$, find the length of the perpendicular from A to BC and hence find the area of triangle ABC.

CHAPTER 9

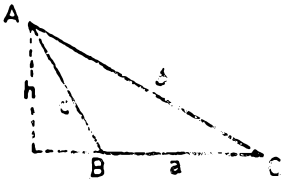
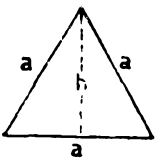
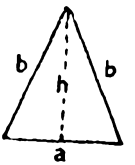
Mensuration—I

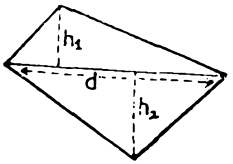
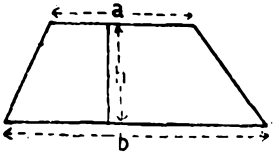
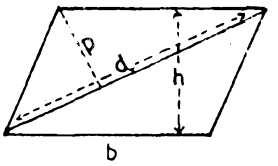
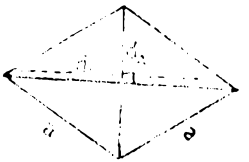
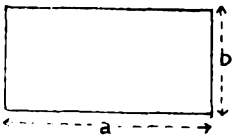
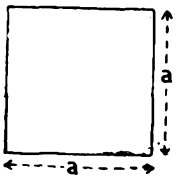
(Polyhedron, Prism and Cylinder)

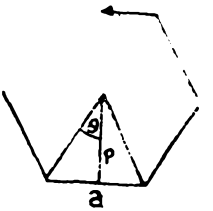
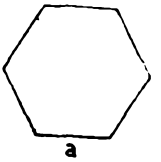
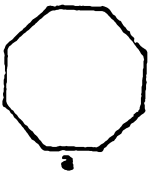
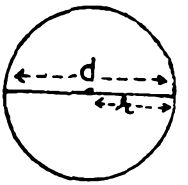
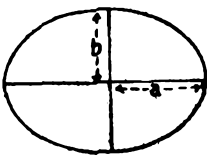
§ 9·1 Mensuration :

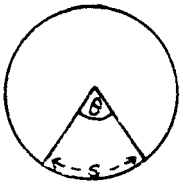
§ 9·11 Plane Mensuration (Revision) :

Plane mensuration deals with finding the lenghs of the sides, perimeters, areas etc. of plane figures such as triangles, quadrilaterals, polygons, circles etc. The formulae, with which you are already familiar, are given in the following table :

	Plane Figure	Area (A)
Triangle with sides of length a, b, c	 $s = \frac{a+b+c}{2}$	Area $= \frac{1}{2} a h$ $= \sqrt{s(s-a)(s-b)(s-c)}$ $= \frac{1}{2} bc \sin A$ $= \frac{1}{2} ca \sin B$ $= \frac{1}{2} ab \sin C$
Equi-lateral triangle	 $h = \frac{a\sqrt{3}}{2}$	$A = \frac{\sqrt{3}}{4} a^2$
Isosceles triangle		$A = \frac{a}{4} \sqrt{4b^2 - a^2}$

	Plane Figure	Area (A)
Quadri-lateral		$A = \frac{1}{2} dh_1 + \frac{1}{2} dh_2$ $= \frac{1}{2} d (h_1 + h_2)$
Trapezium		$A = \frac{1}{2} h (a + b)$
Parallelo-gram		$A = b h$ $= d p$
Rhombus		$A = \frac{1}{2} d_1 d_2$
Rectangle		$A = a b$
Square		$A = a^2$

	Plane Figure	Area (A)
Regular Polygon (n sides)	 $\theta = \frac{180^\circ}{n}$	$A = \frac{1}{2} n a p$ $= \frac{1}{2} n a \sqrt{r^2 - (a/2)^2}$ $= \frac{1}{4} n a^2 \cot \frac{180^\circ}{n}$ $= n p^2 \tan \frac{180^\circ}{n}$ $= \frac{1}{2} n r^2 \sin \frac{360^\circ}{n}$
Regular hexagon $n = 6$		$A = \frac{3a^2\sqrt{3}}{2}$
Regular Octagon $n = 8$		$A = 2a^2 (1 + \sqrt{2})$
Circle	 circumference $= 2\pi r$	$A = \pi r^2$ $= \frac{1}{4} \pi d^2$
Ellipse		$A = \pi ab$

	Plane Figure	Area (A)
Sector of a circle	 $s = r\theta$	$A = \frac{1}{2} r^2 \theta$ $= \frac{1}{2} r s$

§ 9·12. Solid Mensuration (Introduction) :

Solid figures or solids are bounded by plane or curved surfaces and solid mensuration deals with the methods of finding volumes, areas of surfaces etc. of these solids.

Volumes of the solids have three dimensions and their units are cu. cms., cu. meters, cu. inches, cu. ft., etc.

For conversion of units, the following results will be useful.

1 m. = 100 cms.	1 foot = 12 inches
1 sq. m. = 100.100 sq. cms.	1 sq. ft. = 12.12 sq. inches
1 cu. m. = 100.100.100 cu. cms.	1 cu. ft. = 12.12.12 cu. inches.

Exercise 9·1 :

Fill in the blanks :

(1) 5 m. = ——— cms.
(2) 50 cms = ——— m.
(3) 100 sq. cms. = ——— sq. meters.
(4) 4 sq. meters = ——— sq. cms.
(5) 288 sq. inches = ——— sq. ft.
(6) 3 sq. ft. = ——— sq. inches.
(7) 1 sq. yard = ——— sq. ft.
(8) 1 sq. yard = ——— sq. inches.
(9) 1·62 cu. meters = ——— cu. cms.
(10) 1234 cu. cms. = ——— cu. meters.
(11) 2 cu. ft. = ——— cu. inches,
(12) 3456 cu. inches = ——— cu. ft.
(13) 1 cu. yard = ——— cu. ft.
(14) 1 cu. yard = ——— cu. inches.

§ 9·2. Polyhedron :

A polyhedron is a solid bounded by planes. The planes binding the solid are called *faces*. The intersection of any two faces is called an *edge* and the intersection of any two edges is called the *vertex* of the polyhedron.

If all the faces of a polyhedron are congruent regular polygons, then it is called a *regular polyhedron.*

There are only *five* regular polyhedra. They are known as a tetrahedron, a hexahedron (a cube), an octahedron, a dodecahedron and an icosahedron. A *tetrahedron* has 4 faces, a *hexahedron* (a cube) has 6 faces, an *octahedron* has 8 faces, a *dodecahedron* has 12 faces and an *icosahedron* has 20 faces. These are shown in fig. 9·1.

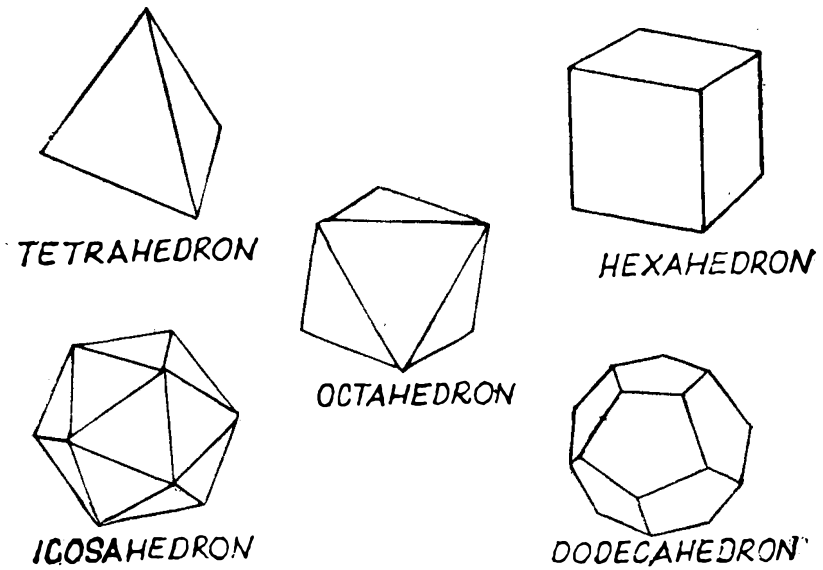


Fig. 9·1

§ 9.5. Prisms :

A polyhedron, of which two faces are polygons in parallel [illegible] and the other faces are parallelograms, is called a *Prism.* A [illegible] in fig. 9·[illegible] Two faces of this prism are ABCDE [illegible] polygons (in this case pentagons)

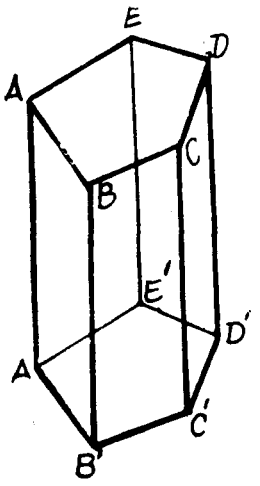


Fig. 9·2

and are in parallel planes. These polygons are known as the *bases* of the prism. ABB'A', BCC'B'....etc. are six parallelograms. These are known as *lateral faces* of the prism. The sum of the areas of the lateral faces is called the *lateral area.* The sum of the lateral area and the areas of the two bases is known as the *total area.* The lines of intersection of the faces are called the *lateral edges.* e. g. for the prism in fig. 9·2. segments AA', BB', CC'....etc. are the lateral edges.

Since the faces are parallelograms, the lateral edges are equal and parallel, e. g. since ABB'A' is a parallelogram $l(AA')=l(BB')$ and $AA' \parallel BB'$ i. e, lateral edges seg AA' and seg BB' are equal and parallel. Similarly the lateral edges seg BB' and seg CC', seg CC' and seg DD'.... etc. are equal and parallel. Hence the *lateral edges of a prism are equal and parallel.*

Further since the faces are parallelograms we have $l(AB)=l(A'B')$ and seg $AB \parallel$ seg $A'B'$, also $l(BC)=l(B'C')$ and seg $BC \parallel$ seg $B'C'$..etc. i.e. the sides of the base ABCDE are congruent and parallel to the corresponding sides of the base A'B'C'D'E'. Hence the *bases of a prism are congruent polygons.*

If the lateral edges are perpendicular to the base, as in the case of the prism in fig. 9·3, the prism is called a *right prism,* otherwise it is called an *oblique prism.* e. g. the prism in fig. 9·2 is an oblique prism whereas fig. 9·3 is of a right prism. The faces of any prism are parallelograms and since in the case of the right prism, the lateral edges are perpendicular to the base, the sides of the parallelogram make angles of 90° with the base and hence the faces are rectangles. Hence *the lateral faces of a right prism are rectangles.*

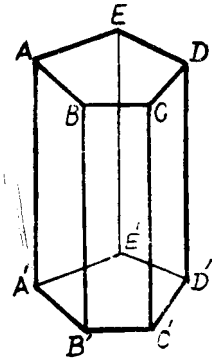


Fig. 9·3

If the bases of a prism are *regular* polygons then the prism is called a *regular prism.*

If the bases of a prism are triangles, it is called a *triangular prism.* If the bases are quadrilaterals, it is called a *quadrangular prism.*

The *altitude* of a prism is the perpendicular distance between the planes of the bases. In case of a right prism, since the lateral edges, e. g. AB, BC .... etc. in fig. 9·3 are perpendicular to the base, the length of each lateral edge can be considered as an altitude. i. e. the perpendicular distance between the planes of the bases is equal to the length of the lateral edge. Hence ' *the altitude of a right prism is equal to its lateral edge.*'

§ 9·31. Section of a Prism :

A *right section* of a prism is a section made by a plane perpendicular to the lateral edges. e. g. the section PQRST in fig. 9·4 and 9·5 are right sections since these planes are perpendicular

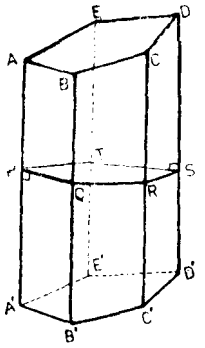


Fig. 9·4

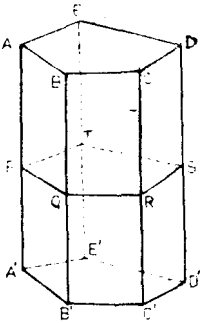


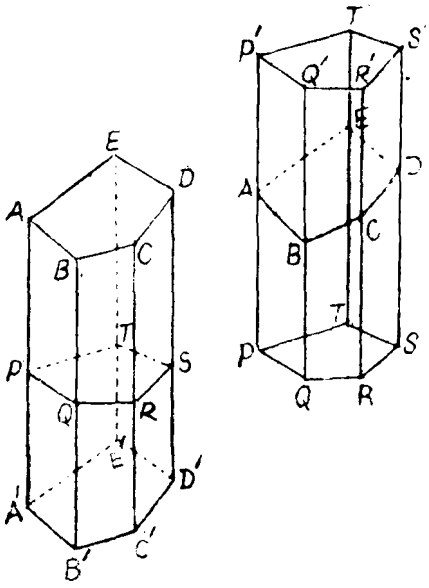
Fig. 9·5

to the lateral edges. Even if the plane is found to be perpendicular to any edge, it will be perpendicular to all the lateral edges. Hence *a plane perpendicular to one lateral edge of a prism is* $\perp$ *to all the lateral edges or to the edges produced.*

Further in case of a right prism, as shown in fig. 9·5, the lateral edges are perpendicular to the two bases and since the lateral edges are also perpendicular to the right section, we can observe that the section PQRST will be parallel to the bases. Not only that, since PQ'B'A', QRC'B', ... etc. are rectangles, the right section PQRST of the right prism will be congruent and parallel to the base. Conversely if a section of a right prism is parallel to its base, then the section is a right section.

In case of any prism, if a plane is drawn parallel to the bace then the edges of the section will be parallel and equal to the corresponding edges of the base because the two parts of each face will be parallelograms. Hence *every section of a prism made by a plane parallel to tts base is congruent to the base.*

In fig. 9·6 (a) we have an oblique prism, where the lateral edges ore not perpendicular to the base. Let PQRST be the right section of this prism. ∴ seg PQ ⊥ seg AA′, seg QR ⊥ seg BB′ and CC′ etc. Now we can cut the part of this prism which is below the right section, i. e. the prism PQRST A′B′C′D′E′ and put it on the top together with the plane ABCDE This is possible, since the bases ABCDE and A′B′C′D′E′ are congruent polygons. The new prism obtained is shown in Fig. 9·6 (b). Let the new position of the plane PQRST be P′Q′R′S′T′. Now since the bases PQRST and P′Q′R′S′T′ of this prism are perpendicular to the lateral edges, this prism PQRST – P′Q′R′S′T′ is a right prism, the volume of which is the same as that of the oblique prism shown in fig. 9·6 (a) i. e. the volume of the oblique prism in fig. 9·6 (a) equals that of the right prism in fig. 9·6 (b) (whose base is the right section of the oblique prism.) Hence, in general, *the volume of an oblique prism equals that of a right prism whose base is a right section of the oblique prism and whose altitude is the lateral edge of the oblique prism.*



(a) Fig. 9 6 (b)

§ 9·32. Lateral area and tolal area of a prism:

(1) Right Prism : In case of a right prism, since the lateral faces are rectangles, the area of a face is given by the product of the length of the corresponding edge of the base and the length of the lateral edge. e. g. for the prism in fig. 9·5 area of the face

AA'B'B is $l(A'B') \cdot l(AA') = l(A'B') \cdot e$, where e is the length of the lateral edge. Similarly the area of the base BB'C' C is $l(B'C') \cdot e$.....

The lateral area = area of AA' B'B + area of BB'C'C + ... etc.

$= l(A'B') \cdot e + l(B'C') \cdot e + l(C'D') \cdot e$....etc.

$= e[l(A'B') + l(B'C') + l(C'D')....]$

$= e$ (perimeter of the base)

(2) Oblique Prism : In case of an oblique prism, the bases are not perpendicular to the edges but the sides of the right section are perpendicular to the edges. Hence area of the face AA'B'B in Fig. 9·4 is $PQ \cdot e$ where e is the length of the edge. Similarly, area of the face BB'C'C = $QR \cdot e$ .....etc. Hence sum of these areas is equal to

$l(PQ) \cdot e + l(QR) \cdot e + l(RS) \cdot e +$ .....etc.

$= e[l(PQ) + l(QR) + l(RS) +$ .....]

$= e$ (Perimeter of the right section)

Total area of a Prism : Total area of a prism = the lateral area + areas of the two bases, where the lateral area can be found as mentioned above.

§ 9·33. Volume of a Prism :

(1) The volume of a *right prism* = area of its base × height and (2) the volume of an oblique prism = area of the right section × lateral edge.

(1) Volume of a right Prism :

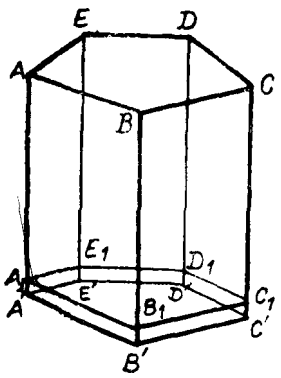


Fig. 9·7

Consider a plane section of the prism at a distance of one unit from the base A'B'C' D' E'.

∴ Volume of the prism A'B'C'D'E' – $A_1B_1C_1D_1E_1$ = Area of base A'B'C'D'E' × 1 unit height.

∴ Volume of the prism ABCDE – A'B'C'D'E'

= Area of base A'B'C'D'E' · h units.

= Area of base × height.

(2) Volume of an oblique Prism :

We have already observed that the volume of an oblique prism equals that of a right prism whose base is a right section of the oblique prism and whose altitude is the lateral edge of the oblique prism.

∴ Volume of an oblique prism
= Volume of the corresponding right prism
= area of base × height
= area of the right section × lateral edge.

Theorem 9·1 : The volumes of two prisms are to each other as the product of their base areas and altitudes.

Proof : Let V_1, V_2 be the volumes of the two prisms; B_1, B_2 be the areas of their bases and h_1, h_2 be the lengths of their altitudes. Then we have to prove that

$$\frac{V_1}{V_2} = \frac{B_1 h_1}{B_2 h_2}$$

But we know that : V_1 = area of base × height = $B_1 h_1$
Similarly V_2 = area of base × height = $B_2 h_2$.

$$\therefore \quad \frac{V_1}{V_2} = \frac{B_1 h_1}{B_2 h_2} .$$

Theorem 9·2 : Two prisms having equal altitudes and having bases of equal area have equal volumes.

Proof : Let V_1, V_2 be the volumes of the two prisms ; B_1, B_2 be the areas of their bases and h_1, h_2 be the lengths of their altitudes.

Since these two prisms have equal altitudes ($h_1 = h_2$) and bases of equal area (*i. e.* $B_1 = B_2$), ∴ $B_1 h_2 = B_2 h_2$. ∴ $V_1 = V_2$.

§ 9·34. Examples :

Ex. 1. The base of a right prism of height 8 cms. is an equilateral triangle of side 3 cms. Find its lateral area, total area and volume.

Ans. Since the base is equilateral triangle of side 3 cms. its area is $\frac{\sqrt{3}}{4} \cdot (3^2) = 9\,\frac{\sqrt{3}}{4}$ sq. cms.

∴ Volume $V = 9 \cdot \frac{\sqrt{3}}{4} \cdot 8 = 18\sqrt{3}$ cubic cms.

Lateral area = (Perimeter of the base) (height)

$= 9 \times 8 = 72$ sq. cms.

Total area $= 72 + 2\left(9\,\frac{\sqrt{3}}{4}\right)$

$= 72 + 9\,\frac{\sqrt{3}}{2} = \frac{1}{2}(144 + 9\sqrt{3})$ sq. cms.

Ex. 2. A right prism has a regular hexagon of side 2 ft. as base and its volume is $48\sqrt{3}$ cu. ft. Find its height and lateral area.

Ans. Area of the base $= \frac{3\sqrt{3}(2)^2}{2} = 6\sqrt{3}$ sq. ft.

Let height be h ft.

$\therefore$ Volume $= (6\sqrt{3} \cdot h)$ cu. ft. $= 48\sqrt{3}$ cu. ft.

$\therefore$ $6\sqrt{3}\,h = 48\sqrt{3}$ $\quad\therefore\ h = 8$ ft.

$\therefore$ Lateral area = (perimeter of the base) (height)

$= 6\,(2)\,(8) = 96$ cu. ft.

Ex. 3. A prism with a rectangular base has a volume of 48 cu. ft. and total area 92 sq. ft. If its height is 8 ft., find the lengths of the sides of its base.

Ans. Let the lengths of sides of its base be x and y feet. Since height is 8′.

Volume $= 8xy$ cu. ft. which is given to be 48 cu. ft.

$\therefore\ 8xy = 48$ $\quad\therefore\ xy = 6$ $\quad\ldots\ (i)$

Similarly, total area

= lateral area + 2 (area of base)

$= 2\,(x+y)\cdot 8 + 2\,(xy)$

$= 2\,(8x + 8y + 6)$ from (i)

But total area is given to be 92 sq. ft.

$\therefore\ 2\,(8x + 8y + 6) = 92$

$\therefore\ 8\,(x+y) + 6 = 46$

$\therefore\ 8\,(x+y) = 40$ $\quad\therefore\ x + y = 5$ $\quad\ldots\ (ii)$

Solving (i) and (ii), $x = 3$, $y = 2$.

$\therefore$ The sides of the base are 3′ and 2′.

Ex. 4. Two similar prisms have polygonal bases whose sides are in the ratio 1 : 2. If their heights are in the ratio 1 : 5. Find the ratios of their lateral areas and volumes.

Ans. Since ratio of sides of the base is 1 : 2 and ratio of heights = 1 : 5

$\therefore$ Ratio of lateral areas $= \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$ i. e. 1 : 10

and ratio of volumes $= \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{5}\right) = \frac{1}{20}$ i. e. 1 : 20.

Exercise 9.2 :

(1) The base of a right prism 4 metres high is an equilateral triangle whose side is 1 metre. Find the lateral area and the volume.

(2) The base of a right prism is a triangle ABC right angled at C. If l (AC) = 3 mts., l (AB) = 5 mts. and height of the prism is 20 metres, find the volume and lateral area of this prism.

(3) A quadrilateral ABCD is the base of a prism whose height is 20 cms. If l (BC) = 7 cms, l (CD) = 24 cms and l (DA) = 20 cms and $m\angle A = m\angle C = 90°$, find the volume of the prism.

(4) A regular hexagon is the base of a prism. Every edge of the prism is of length 2 m. Find the volume of the prism.

(5) The base of a prism of height 12 ft. is a trapezium, the lengths of whose parallel sides are 7 ft. and 23 ft. and the distance between them is 10 ft. Find the volume of the prism.

(6) The volume of a triangular prism is 90 cub. cms. and the lengths of the sides of the base are 12 cms, 13 cms and 5 cms. Find the height of the prism and find its lateral area.

(7) The base of a right prism is a regular pentagon whose side is 10 cm. If the height of the prism is 100 cms, find the area of its lateral surface.

(8) What weight of water will fill a vessel in the form of a prism whose base is a regular hexagon of side 2 ft. and whose height is 5 ft. ? (weight of 1 cu. ft. of water is 62·5 lbs.)

(9) Compare the lateral surfaces and the volumes of the following right prisms :

(i) base is an equilateral triangle of side 6 cms. and height is 10 cms; base is an equilateral of side 12 cms, and height is 20 cms.

(ii) base is a rhombus of side 3 cm and height is 8 cm; base is a rhombus of side 9 cm and height is 16 cms.

(iii) base is a regular hexagon of side 5″ and height is 1 ft.; base is a regular hexagon of side 10″ and height is 5 ft.

(10) V_1 and V_2 are the volumes of two right prisms, whose heights are h_1 and h_2 and bases are regular polygons with areas B_1 and B_2 respectively. If $\frac{a_1}{a_2} = \frac{2}{3} = \frac{h_1}{h_2}$ where a_1, a_2 are the lengths of

the sides of the base, find $V_1 : V_2$ and the ratio of their lateral surfaces.

(11) The volume of a regular hexagonal prism is $60\sqrt{3}$ cu. cms. and its lateral area is 120 sq. cms. Find the lengths of the edges and the height.

(12) The total surface area of a right prism with a rectangular base is 208 sq. m. and its height is 8 m. If the volume of this prism is 192 cu. m., find the lenghts of the side of its base.

§ 9·35. A Parallelopiped :

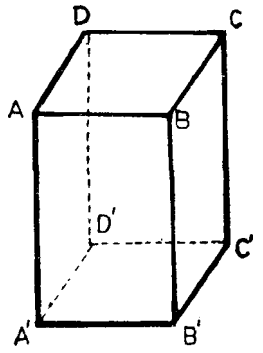


Fig. 9·8

A parallelopiped is a prism whose bases are *parallelograms*, It may be right or oblique. Since a parallelopiped is a prism, we can use the formulae for the lateral area and volume of a prism (right prism or oblique prism) for finding the lateral area and volume of a parallelopiped which we rewrite here.

Right parallelopiped :

Lateral area = Height. × Perimeter of the base

Total area = Lateral area + areas of the two bases.

Volume = Area of the base × height.

Oblique parallelopiped :

Lateral area = Lateral edge × Perimeter of the right section.

Total area = Lateral area + areas of the two bases.

Volume = Area of the base × height.

In case of a right parallelopiped the faces will be rectangles, but base is a parallelogram, not a rectangle. But if the base of a right parallelopiped is also a rectangle, then it is known as a rectangular parallelopiped or a rectangular solid or a cuboid.

Hence a rectangular parallelopiped is solid bounded by three pairs of parallel rectangular facec. i. e. all the six faces are rectangles.

§ 9·36. Rectangular Parallelopiped :

ABCD and A'B'C'D', ABB'A' and DCC'D', and BB'C'C and AA'D'D are the three pairs of parallel rectangular faces. Thus

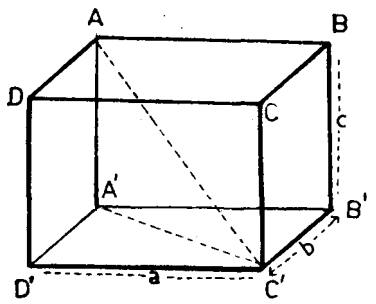


Fig. 9·9

there are six faces and all of them are rectangles. Segments AB, BC, CD, AD, AA', BB'.. ..... etc. are the twelve edges of this solid. Let a, b, c be the length, breadth and the height of this solid. Then the volume of the rectangular solid is abc. We can consider A'B'C'D' as the base. Its area is ab and height is c.

$\therefore$ Volume = area of the base·height

$$= (ab)\,c = abc.$$

The areas of the rectangles ABCD and A'B'C'D' are ab sq. units each, of the rectangles ABB'A' and DCC'A' are ca sq. units each and of the rectangles BB'C'C and AA'D'D are bc sq. units each. Hence the sum of the area of these six rectangles is

$$2ab + 2bc + 2ca = 2\,(ab + bc + ca)$$

$\therefore$ *Total surface area* of a rectangular solid with sides a, b, c is $2\,(ab + bc + ca)$.

Theorem 9·3 : The square on the diagonal of a rectangular parallelopiped is equal to the sum of the square of the lengths of it sconcurrent edges.

Proof: Let ABCD – A'B'C'D' be the rectangular Parallelopiped, the lengths of whose edges are a, b, c as shown in fig. 9·10.

$\therefore$ AC', BD', and B'D are the diagonals of the rectangular parallelopiped. Let d be the length of a diogonal AC'. Join A'C'.

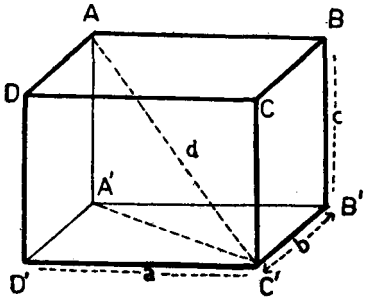


Fig. 9·10

Naw A'C'D' is a right angled triangle.

$\therefore [l(A'C')]^2 = [l(A'D')]^2 + [l(C'D')]^2 = a^2 + b^2$ .... (i)

AA'C' is also a right angled triangle. $\therefore [l(AC')]^2$

$= [l(AA')]^2 + [l(A'C')]^2$

$= c^2 + [(lA'C')]^2$

$= c^2 + (a^2 + b^2)$ by (1)

$= a^2 + b^2 + c^2$

Similarly $[\, l\,(BD\,)\,]^2 = a^2 + b^2 + c^2$ and $[\, l\,(B'D)\,]^2$
$= a^2 + b^2 + c^2$

$\therefore$ Square on the diagonal of a rectangular parallelopiped with sides a, b, c is $(\, a^2 + b^2 + c^2\,)$.

§ 9·37. **Cube:**

When all the faces of a rectangular solid are squares, then the solid is called a *cube.* (see fig. 9·11.) Hence in case of a cube $a = b = c$.

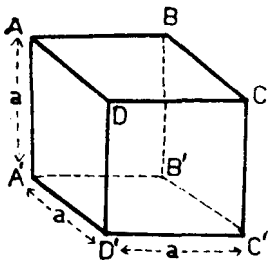


Fig. 9·11

$\therefore$ Volume of a cube with side a $= a \cdot a \cdot a = a^3$.

Each of the six squares has an area a^2.

$\therefore$ Total surface area of a cube with side a is $6a^2$.

§ 9·38. **Examples:**

Ex. 1. Find the volume of an iron rod of length 10 cms. and whose right section is a square of side 5 cms. If one cu. cm of iron weights 7·027 gms., find the weight of the rod.

Ans. The section of the rod is a square whose sides is 5 cms. Hence the area of this section is 25 sq. cms.

$\therefore$ Volume of the rod = area of the right section $\times$ length
$= 25 \times 10$
$= 250$ cu. cms.

$\therefore$ The weight of the rod $= 7{\cdot}027 \times 250 = 1756{\cdot}75$ gms.

Ex. 2. The length, breadth and height of a box are 24 cms, 8 cms, 6 cms. respectively. Find the length of the longest rod that can be placed in the box.

Ans. The length of the longest rod
= the length of the diagonal
$= \sqrt{(\text{length})^2 + (\text{breadth})^2 + (\text{height})^2}$
$= \sqrt{24^2 + 8^2 + 6^2}$
$= \sqrt{576 + 64 + 36}$
$= \sqrt{676} = 26$ cms.

Ex. 3. A box without a lid is made of wood 1 cm. thick. It measures 10 cms. in length, 7 cms in breadth and 5 cms. in

height. If it is to be painted on the surfaces, how many superfical sq. cms. of paint will be required ?

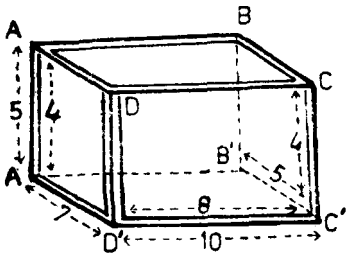


Fig. 9·12

Here external length = 10 cms.
external breadth = 7 cms.
and external height = 5 cms.
Thus internal length = 10 − 2
= 8 cms.
internal breadth = 7 − 2
= 5 cms.
and internal height = 5 − 1
= 4 cms.

Hence the area that is painted is calculated as follows :

Area of four external walls = $2 \times 5 (10 + 7) = 170$ sq. cms (i)

Area of four internal walls = $2 \times 4 (8 + 5) = 104$ sq. cms. (ii)

Area of inside floor = $8 \times 5 = 40$ sq. cms. .... (iii)

Area of outside bottom = $10 \times 7 = 70$ sq. cms. .... (iv)

Area of the top of the four walls at the open end = area of outside bottom − area of inside floor = 70 − 40 sq. cms. = 30 sq. cms. .... (v)

Adding (i), (ii), (iii), (iv) and (v) we get the total area to be painted = 170 + 104 + 40 + 70 + 30
= 414 sq. cms.

Hence 414 sq. cms. of paint will be required.

Exercise 9·3 :

(1) One end of a cord is tied to a nail hammered in the centre of the floor of a room measuring 12 metres long, 4 metres wide and 3 metres high. The other end of the cord is tied to the leg of a sparrow. What should be the minimum length of the cord which would enable the sparrow to reach every point in the room ?

(2) Find the volume and the surface area of a slab of metal whose length, breadth and thickness are 10 mts. 5 mts. and 10 cms.

(3) The diagonal of cube is 10 cms. Find its volume and surface area.

(4) The diagonal of the base of a cube is 10 cms. Find the area of its surface and its volume.

(5) Three cubes of metal whose edges are 6, 8 and 10 cms. respectively, are melted and formed into a single cube. Determine the edge of the new cube assuming no wastage of metal in the process.

(6) A cube whose each edge is 20 cms. long has on each of its faces a circle of 20 cms. diameter painted black. What is the total unpainted surface left out ?

(7) If V is the volume of a cube and d is the length of its diagonal show that $d^3 = 3\sqrt{3}\,V$.

(8) If S_1, S_2 are the surface areas and V_1, V_2 are the volumes of two rectangular solids with sides x, y, z and $3x, 3y, 3z$ respectively, find the ratios $S_1 : S_2$, $V_1 : V_2$.

(9) A box is in the form of a rectangular parallelopiped. Its height exceeds its breadth by 6 ft. and its length exceeds its height by 4 ft. If the volume of the parallelopiped be 351 cubic feet, find the lengths of the sides of the parallelopiped.

(10) A box, without a lid and made of wood 1 cm. thick, is to be painted inside and outside. If it measures 20 cms in length 10 cms in breadth and 6 cms in height, how many superficial sq. cms. of paint will be required for each coat ?

(11) A ditch 1 metre wide is dug outside a plot 8 mts. by 10 mts. and the eart removed to raise the level of the plot by 2 cms. Find the depth of the ditch.

(12) A school room is to be built to accommodate 70 students so as to allow 8·5 sq. metres of floor and 110·5 cubic metres of space for each student. If the length of the room is to be 34 metres, find out the breadth and height.

(13) A rectangular metal box without a lid measures externally 60 cm. long, 45 cms wide and 30 cms deep and the material has a uniform thickness of 2·5 cms. If the metal costs Rs. 2·50 per 100 cubic cms and the making charge be 10 per cent of the cost of the material, find to the nearest naya paisa the cost of the box.

(14) Find how many bricks, of which the length, breadth and thickness are 9, 4·5 and 3 inches, will be required to build a wall of which the length, height, and thickness are 72, 8, and 1·5 ft.

§ 9·4. Cylinder :

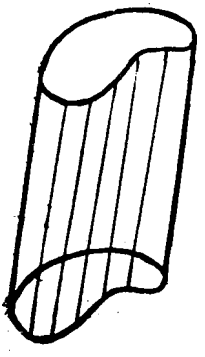


Fig. 9·13

A *cylinder* is a solid generated by a straight line which always moves parallel to itself round the boundary of any curve and is bounded by two parallel planes. (fig. 9·13). The curve is called the *guiding curve* of the cylinder. Any position of the moving line, is called a *generator* of the moving surface. The generators of a cylinder are always parallel and equal in length.

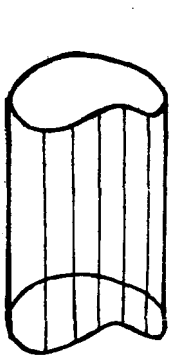


Fig. 9·14

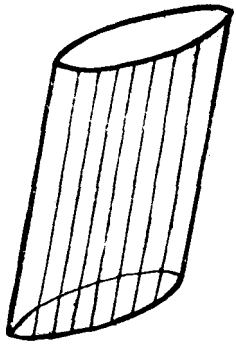


Fig. 9·15

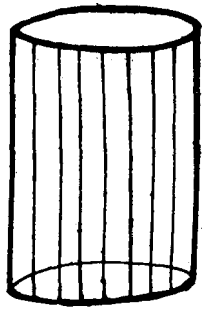


Fig. 9·16

If the generator is perpendicular to the base it is called a *right cylinder* (fig. 9·14). If the guiding curve is a circle, the cylinder is called a *circular cylinder* (fig. 9·15) and further if the generator is perpendicular to the base, the circular cylinder is called a *right circular cyiinder* (fig. 9·16).

The bases of a circular cylinder are equal circles and a right section of a right circular cylinder is also a circle but a right section of any circular cylinder may not be a circle.

The line joining the centres of the circular bases is called the *axis* of the circular cylinder. The axis of a circular cylinder is parallel and equal to the generator.

Since a cylinder may be considered as the limiting case of an inscribed prism as n, the number of faces increase indefinitely in such a way that the length of each edge tends to zero, all the formulae of right prism can be applied to right cylinder and of oblique prism to oblique cylinder.

§ 9·41. Volume and lateral area of a right circular cylinder :

Theorem 5·4 : (1) The volume of a right circular cylinder is equal to the product of the area of its base and the altitude.

(2) The lateral area of the right circular cylinder is the product of the perimeter of its base and the altitude.

Proof: Let V be the volume and S be the lateral surfaces of a right circular cylinder whose altitude is h and the area of whose base is B. Let P be the perimeter of the base.

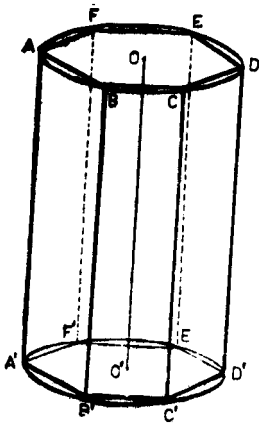


Fig. 9·17

Let a prism, with n sided regular polygons as bases, be inscribed in the given right circular cylinder, as shown in fig. 9·17. Hence the altitude of the prism is also h. Let B_n be the area of the base of the pirsm and let P_n be the perimeter of the base.

Hence the volume V_n of the prism is $V_n = B_n\, h$ end the area S_n of its lateral surface is $P_n\, h$.

As n increases indefinitely, the inscribed prism approaches ihe right circular cylinder.

Hence as $n \to \infty$:

the inscribed prism → the right circular cylinder,

the polygonal base B_n → the circular base B,

the perimeter P_n of the base B_n → the circumference of the circular base,

the lateral surface of the inscribed prism] → [the lateral surface of the right circular cylinder,

the volume of the inscribed prism) → (the volume of the right circular cylinder.

i. e. $S_n \to S$ and $V_n \to V$ as $n \to \infty$.

∴ Volume of the cylinder

$$= \lim_{n\to\infty} (\text{volume of the inscribed prism})$$

$$= \lim_{n\to\infty} V_n = \lim_{n\to\infty} B_n\, h = h \lim_{n\to\infty} B_n$$

$= h\,B$ (∵ as $n \to \infty$, $B_n \to B$)

$= B\,h =$ (area of the circular base of the right circular cylinder)·(altitude) .... (i)

Lateral area S of the right circular cylinder

$$= \lim_{n\to\infty} (\text{Lateral area } S_n \text{ of the inscribed prism})$$

$$= \lim_{n\to\infty} (\text{Perimeter of base of the prism})(\text{altitude})$$

$$= \lim_{n\to\infty} P_n h = h \cdot \lim_{n\to\infty} P_n = h\,P = P\,h$$

= (Perimeter of the circular base of the right circular cylinder)·(altitude) ..... (ii)

Theorem 9·5: The volume of a right circular cylinder with base of radius r and altitude h is $\pi r^2 h$.

Proof: The base of the given right circular cylinder is the circle with radius r. Hence the area of this base (circle with radius r) is πr^2.

$\therefore$ Volume of the given cylinder

= area of base × altitude

= (area of the circle with radius r)·$h = \pi r^2 h$.

Theorem 9·6: The lateral area of a right circular cylinder with base of radius r and altitude h is $2\pi rh$.

Proof: The base of the given right circular cylinder is the circle with radius r. Hence the perimeter of the base is the circumference of the circle, whose radius is r, i. e. $2\pi r$.

$\therefore$ Lateral area of the given right circular cylinder.

= perimeter of the base × altitude

= (circumference of the circle with radius r)·(altitude)

$= 2\pi rh$.

Theorem 9·7: Total area of a right circular cylinder, whose base is of radius r and altitude is h, is $2\pi r(h+r)$.

Proof: By theorem 9·6, the lateral area of the given cylinder is $2\pi rh$ and each base (circle) is of area πr^2.

$\therefore$ Total area = lateral area + sum of the areas of the two bases

$$= 2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2$$

$$= 2\pi r(h+r).$$

§ 9·42. Similar Circular Cylinders :

Two circular cylinders are said to be *similar* if the lengths of their generating lines and the radii of their bases are proportional i. e. $\frac{l'}{l} = \frac{r'}{r}$, where l, l' are the lengths of the generating lines and r, r' are the radii of their bases.

Theorem 9·8 : The surface areas of two similar right circular cylinders are to each other as the squares of the corresponding dimensions.

Proof : Consider two similar right circular cylinders whose generating lines (i. e. altitudes) have lengths h and h_1 and the radii of whose bases are r and r_1 respectively. Let their lateral surface areas be S and S_1 respectively. Hence we have to show that

$$\frac{S}{S_1} = \frac{h^2}{h_1^2} = \frac{r^2}{r_1^2}.$$

Now since the cylinders are similar, $\frac{h}{h_1} = \frac{r}{r_1}$

$$\therefore \quad \frac{S}{S_1} = \frac{2\pi rh}{2\pi r_1 h_1} = \frac{r}{r_1} \cdot \frac{h}{h_1}$$

Since $\frac{r}{r_1} = \frac{h}{h_1}$, we have $\frac{S}{S_1} = \frac{r^2}{r_1^2} = \frac{h^2}{h_1^2}$ as required.

Theorem 9·9 : The volumes of two similar right circular cylinders are to each other as the cubes of their corresponding dimensions.

Proof : Consider two similar right circular cylinders whose generating lines (i. e. altitudes) have lengths h and h_1 and the radii of whose bases are r and r_1 respectively. Let their volumes be V and V_1 respectively. We have to prove that

$$\frac{V}{V_1} = \frac{h^3}{h_1^3} = \frac{r^3}{r_1^3}$$

Since the cylinders are similar, $\frac{h}{h_1} = \frac{r}{r_1}$ .... (i)

Also $\frac{V}{V_1} = \frac{\pi r^2 h}{\pi r_1^2 h_1} = \left(\frac{r}{r_1}\right)^2 \cdot \left(\frac{h}{h_1}\right) = \frac{r^3}{r_1^3} = \frac{h^3}{h_1^3}$ as required, using (i).

§ 9·43. Examples:

Ex. 1. A concrete roller 4 metres long and 3·5 metres in diameter makes 400 complete revolutions to cover a path. Find the area of the path. $\left(\text{Take } \pi = \frac{22}{7}\right)$.

Ans. The area of the lateral surface $= 2\pi rh$

$$= 2 \cdot \frac{22}{7} \cdot \frac{1{\cdot}75}{1} \cdot 4 = (176)\,(0{\cdot}25) = 44{\cdot}00 \text{ sq. metres.}$$

$\therefore$ The area covered in 400 complete revolutions

$= 400\ (44)$ sq. metres $= 17600$ sq. metres.

Ex. 2. A metal rod sells at Rs. 100 per kg. If the weight of the metal per cubic cms. be 11·34 gms, find its price per metre, being given that the rod has a cross-section of diameter 2 cms. $\left(\text{Take } \pi = \frac{22}{7}\right)$.

Ans. The area of cross-section of rod $= \pi r^2$

$$= \frac{22}{7} \cdot (1)^2 \text{ sq. cms.}$$

$\therefore$ The volume of the rod = (area of cross section)·(length)

$$= \frac{22}{7} \cdot 100 = \frac{2200}{7} \text{ cu. cms.}$$

The weight of the metal in the rod $= (11{\cdot}34) \cdot \frac{2200}{7}$

$= 3564$ gms. $= 3{\cdot}564$ kgs.

$\therefore$ Price of metal per metre $= (3{\cdot}564 \times 100)$ rupees

$=$ Rs. 356·4 per metre.

Ex. 3. Silver wire of diameter 5 mm. and length 3 metres is drawn to a length of 3·75 metres. Find the new radius.

Ans. The volume of the wire $= \pi r^2 l$ (where l is the length)

$= (\cdot 25)^2 \cdot 300\ \pi$ cu. cms.

Let r_1 be the new radius of drawn wire

$\therefore$ The volume of the wire $= \pi r_1^2\ (375) = 375\ r_1^2\ \pi$ cu. cms.

$\because$ The volume of the wire remains the same.

$\therefore$ Initial volume = final volume

$\therefore$ $(0{\cdot}25)^2 \cdot 300\pi = 375\ r_1^2\ \pi$

$\therefore$ $r_1^2 = \frac{(0{\cdot}0625) \cdot 300}{375}$ $\therefore$ $r_1 = \frac{2}{100}\sqrt{125} = \frac{\sqrt{5}}{10}$ cms.

Second Method :

Let V_1, r_1 and l_1 be the initial volume, initial radius and initial length. Let V_2, r_2 and l_2 be the final volume, final radius and final length.

Now $V_1 \propto r_1^2 l_1$

$V_2 \propto r_2^2 l_2 \quad \therefore \quad \frac{V_1}{V_2} = \frac{r_1^2 l_1}{r_2^2 l_2} = 1$

$\therefore \quad (2{\cdot}5)^2 \cdot 300 = r_2^2 \cdot 375$

$\therefore \quad r_2^2 = \frac{(0{\cdot}25)^2 \cdot 300}{375} \quad \therefore \quad r_2 = \frac{\sqrt{5}}{10}$ cms.

Ex. 4. The inner and outer radii of a hollow cylinder are 15 and 19 cms. respectively. If this cylinder be melted and formed into a solid cylinder of the same length, find the new radius.

Ans. The volume of the given hollow cylinder

$= \pi l (r_2^2 - r_1^2)$ cu. cms., where r_2 = outer radius, r_1 = inner radius.

$= \pi l (19^2 - 15^2)$ cu. cms.

$= \pi l \ (19 + 15)(19 - 15)$ cu. cms.

$= \pi l (34)(4)$ cu. cms.

$= 136 \pi l$ cu. cms.

Let r be the new radius of solid cylinder

$\therefore$ Its volume is $\pi r^2 l = 136 \pi l$.

$\therefore \quad r^2 = 136 \quad \therefore \quad r = \sqrt{136} = 2\sqrt{34}$ cms.

Ex. 5. If a round pillar 7 inches in diameter contains 2310 cubic inches of stone. What is the diameter of a pillar of equal length which contains ten times as much. $\left(\text{Take } \pi = \frac{22}{7}\right)$.

Ans. Volume of round pillar $= \frac{22}{7} \cdot \frac{49}{4} \cdot h = 2310$

$\therefore \quad h = \frac{2310 \times 7 \times 4}{22 \times 49} = 60''$.

Volume of new pillar which contains 10 times the previous value = 23100 cu. inches.

If its radius is r then $23100 = \frac{22}{7} \cdot r^2 \cdot 60$

$\therefore \quad r^2 = \frac{23100 \times 7}{22 \times 60} \quad \therefore \quad r = \frac{7\sqrt{10}''}{2} \quad \therefore$ Diameter $= 7\sqrt{10}''$.

Exercise 9·4 :

(1) How many cubic metres of earth must be dug out to sink a well 35 metres deep and 4 metres in diameter?

(2) A roller is made of iron 4 inches thick. It is 6 feet long and has an outside diameter of 4 feet. Find its weight, given that 1 cubic foot of iron weighs 420 lbs. How many revolutions will it have to make to cover a rectangular track 176 ft. by 6 ft.?

(3) Find the weight of a cast iron pipe whose length is 105 cms., the bore 3 cms. and the thickness of the metal 1 cm. (One cubic cm. of cast iron weighs 1/4 lb.)

(4) A well is to be dug with inside diametre 2 metres and 14 metres in depth. Find the quantity of earth to be excavated and the quanttity of brickwork required for a lining of 20 cms. around the circumference.

(5) The weight of a cube of edge 2 inches is 4·06 lbs. Find the weight of a cylinder, of the same material, of height 10 inches with a base diameter of 4 inches.

(6) A sheet of copper 11 cms. long, 7 cms. wide and 2 cms. thick is drawn into a wire of uniform thickness 100 metres long; find the diameter of the wire.

(7) A hollow metallic cylindrical drum has an external diameter of 45·5 cms. and length 60 cms. The hollow space inside is a cylinder of diameter 35 cms. and length 52 cms. Calculate the weight of the drum when empty and when full of water, if the metal of which the drum is made weighs 6·8 gms. per cc. and 1 litre of water weighs 1 kg.

(8) A rectangular sheet of paper 22 in. long and 12 in. broad can be curved so as to form the lateral surface of two different right circular cylinders. Find the difference between the volumes of the two different cylinders which can be so formed.

(9) The area of the base of a right circular cylinder is 100π sq. inches and the volume 5 cu. ft. Find the area of the curved surface.

(10) A cylinder whose base is a circle is circumscribed about a right prism of altitude 12·6 feet. Find the volume of the cylinder if the base of the prism is (i) a square of side 3 feet, (ii) a rectangle 3 feet by 4 feet (iii) an equilateral triangle of side 4 feet.

CHAPTER 10

Mensuration—II

(Pyramid, Cone and Sphere)

§ 10·1. Pyramid :

A polyhedron of which one face is a polygon and the other faces are tringles having a common vertex at some point which is not in the plane of the polygon is called a *Pyramid.*

The polygon is called the *base* and the triangles are called the *lateral faces* of the pyramid. An intersecting line of two faces is called a lateral edge. The lateral edges meet at the *vertex.* The length of the perpendicular from the vertex to the base is called the *altitude* or the *height* of the pyramid.

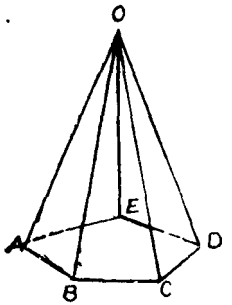


Fig. 10.1

A pyramid is shown in fig. 10·1. O is the vertex. The polygon ABCDE is the base. The triangles OAB, OBC, OCD, ODE and OEA are the lateral faces. Segments OA, OB, OC, OD, OE are lateral edges.

If the base of a pyramid is a triangle, the pyramid is called a triangular pyramid or *tetrahedron.* If the base is a quadrilateral, the pyramid is called a *quandrangular pyramid.* Similarly, a pentagonal pyramid has a pentagon as the base and a hexagonal pyramid has a hexagon as the base.

§ 10·11. Regular Pyramid :

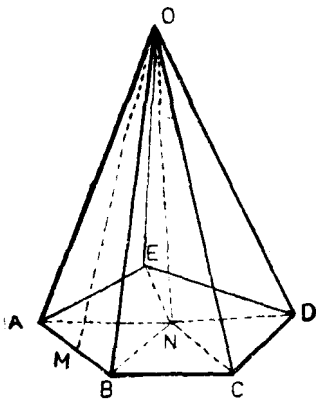


Fig. 10·2

If the base of a pyramid is a regular polygon such that the altitude from the vertex meets the base at its centre then the pyramid is called a *regular pyramid.*

A regular pyramid is shown in Fig. 10·2. N is the centre of the base polygon, hence the lengths of AN, BN, CN, DN, EN are equal. Further ON is the altitude, hence $m\angle ONA = m\angle ONB = \ldots = 90°$. $\therefore$ The triangles OAN, OBN. OCN ODN and OEN are congruent triangles. Hence $l(OA) = l(OB) = l(OC) = l(OD) = l(OE)$, i. e. *the lateral edges of a regular pyramid are equal in length.*

Each lateral face of a regular pyramid contains two lateral edges which are equal in length. Hence, the lateral faces are *isosceles triangles.* Also since such face contains one side of the base and all the sides of polygonal base are equal, all the lateral faces are *congruent.* Hence, *the lateral faces of a regular pyramid are congruent isosceles triangles.*

Since all the faces are congruent triangles, their altitudes drawn from the vertex are also of equal length.

The altitude of any lateral face is called the *slant height* of the regular pyramid.

§ 10·12. Lateral area and volume of a pyramid:

Theorem 10·1. The lateral area of a regular pyramid is equal to one-half the product of its slant height and the perimeter of the base.

Proof: Let O be the vertex and ABCDE .... be the base of a regular pyramid. Let l be its slant height. In triangle OAB, draw OM perpendicular to AB.

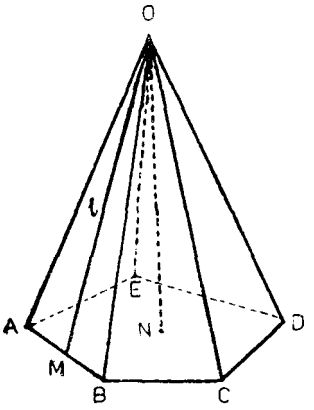


Fig. 10·3

$\therefore$ $l(\text{OM}) = l$

Area of triangle OAB

$= \frac{1}{2}$ base $\times$ height

$= \frac{1}{2}\, l(\text{AB}) \cdot l$

Similarly, the areas of the triangles OBC, OCD, ODE, ... etc. are $\frac{1}{2}\, l(\text{BC}) \cdot l$, $\frac{1}{2}\, l(\text{OD}) \cdot l$, $\frac{1}{2}\, l(\text{DE}) \cdot l$ .... etc.

$\therefore$ Lateral area = sum of the areas of the lateral faces

$= \frac{1}{2}\, l(\text{AB}) \cdot l + \frac{1}{2}\, l(\text{BC}) \cdot l + \frac{1}{2}\, l(\text{CD}) \cdot l + \ldots$

$= \frac{1}{2}\, l\,[l(\text{AB}) + l(\text{BC}) + l(\text{CD}) + \ldots]$

$= \frac{1}{2} \cdot$ slant height $\cdot$ perimeter of the base.

Theorem 10·2: The volume of a triangular pyramid is equal to one-third the product of the area of its base and its altitude.

Proof: Let A—BCD be the given triangular pyramid whose altitude is h and the area of the base BCD is B_1.

Complete a prism AEF – BCD whose lateral edges seg FD and seg EB are equal and parallel to seg AC. Join DE.

Now the prism can be considered as made of three triangular pyramids, viz. A – BCD, A – DEF and A–BDE.

Now, since the bases of the prism viz. triangle AEF and tri-

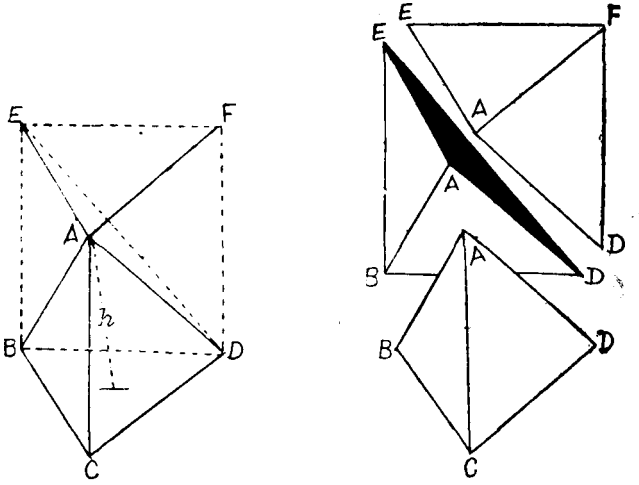


Fig. 10·4

angle BCD are equal in area, the pyramids A – BCD and D–AEF (i. e. A – DEF) have equal bases and equal heights. Hence the volumes of pyramids A – BCD and A – DEF are equal .... (i)

Further the triangles DEF and BDE are also equal in area, ence the volumes of the pyramids A – DEF and A – BDE area also equal .... (ii)

From (i) and (ii) we find that volumes of the 3 triangular pyramids, considered as the part of the prism AEF – BCD are equal. Hence the volume of the prism AEF – BCD is three times of the volume of the pyramid A – BCD.

$\therefore$ Volume of the pyramid A – BCD

$$= \frac{1}{3} \text{ volume of the prism AEF} - \text{BCD}$$

$$= \frac{1}{3} \cdot \text{area of the base BCD} \cdot \text{height } h$$

$$= \frac{1}{3} \cdot B_1 \cdot h$$

Theorem 10·3 : The volume of any pyramid is equal to one-third the product of the area of its base and the altitude.

Proof : Let O – ABCDE..... be any pyramid, with area of the base S and altitude h.

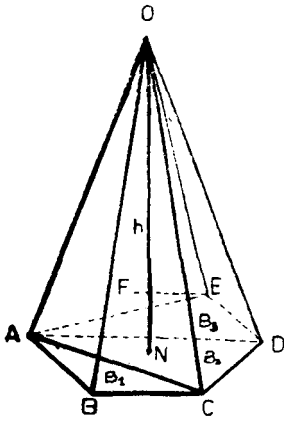


Fig. 10·5

Since any pyramid can be divided into a number of triangular pyramids such as O – ABC, O – ACD, O– ADE ... etc. of height h and with areas of bases $B_1, B_2, B_3 \ldots$ such that

$$B_1 + B_2 + B_3 + \ldots = S$$

$\therefore$ Volume of the pyramid O – ABCDE.... = sum of the volumes of the pyramids O – ABC, O – ACD, O – ADE....etc.

$$= \frac{1}{3} \cdot B_1 h + \frac{1}{3} \cdot B_2 \cdot h + \frac{1}{3} \cdot B_3 \cdot h + \ldots.$$

$$= \frac{1}{3} \cdot h \cdot (B_1 + B_2 + B_3 \ldots.)$$

$$= \frac{1}{3} \cdot h \cdot S = \frac{1}{3} \cdot S \cdot h = \frac{1}{3} \text{ (area of the base)} \times \text{(altitude)}$$

§ 10·13. Section of a pyramid :

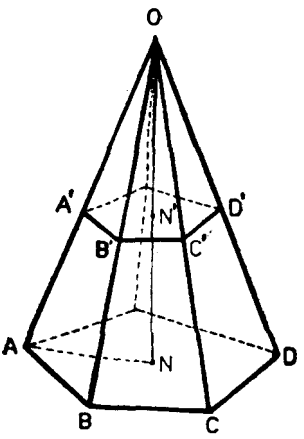


Fig. 10·6

The portion of the pyramid included between its base and a section parallel to the base is called a *frustum* of a pyramid.

The pyramid O – ABCDE..... shown in Fig. 10·6 is cut by a plane A'B'C'D'E'....drawn parallel to the base ABCDE....Then the solid bounded between the plane ABCDE ... , the plane A'B'C'D'E' .... and the lateral faces is called the frustum of the pyramid O – ABCDE.... Let seg ON be the altitude of the pyramid intersecting A'B'C'D'E' at N'. In triangle OAN, seg A'N' ∥ seg AN

$$\therefore \quad \frac{l(ON')}{l(ON)} = \frac{l(OA')}{l(OA)} = \frac{l(OB')}{l(OB)}$$

Similarly, $\quad \frac{l(ON')}{l(ON)} = \frac{l(OC')}{l(OC)} = \frac{l(OD')}{l(OD)} = \ldots$

$$\therefore \quad \frac{l(ON')}{l(ON)} = \frac{l(OA')}{l(OA)} = \frac{l(OB)}{l(OB)} = \frac{l(OC')}{l(OC)} = \ldots$$

Hence the lateral edges and the altitude are divided proportionally .... (i)

Further in triangle OAB, seg A'B || seg AB

$$\therefore \quad \frac{l(OA')}{l(OA)} = \frac{l(OB')}{l(OB)} = \frac{l(A'B')}{l(AB)}$$

Similarly, $\frac{l(OB')}{l(OB)} = \frac{l(OC')}{l(OC)} = \frac{l(B'C')}{l(BC)}$ ....etc.

$$\therefore \quad \frac{l(A'B')}{l(AB)} = \frac{l(B'C')}{l(BC)} = \frac{l(C'D')}{l(CD)} \text{ ....etc.}$$

Hence the section formed is similar to the base .... (ii)

From (i) and (ii) we have the result : *"If a pyramid is cut by a plane parallel to its base, the lateral edges and altitude are divided proportionally and the section formed is similar to its base"*.

We assume, without proof, the following result: " The volume of the frustrum of a pyramid of height H and the area of whose bases are S_1 and S_2 is $\frac{1}{3} H (S_1 + S_2 + \sqrt{S_1 S_2})$.

Theorem 10·4 : The areas of parallel sections of the pyramid are to each other as the squares of the distances from the vertex.

Proof : Let ABCDE.... and A'B'C'D'E'....be the parallel sections of a pyramid at distances h and h' ($h > h'$) from the vertex O of the pyramid. Draw ON perpendicular to the plane ABCDE....intersecting the plane A'B'C'D'E'.... at N'.

$\therefore$ $l(ON) = h$ and $l(ON') = h'$.

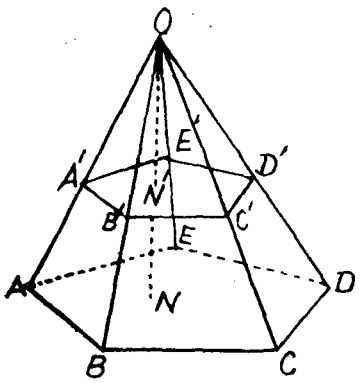


Fig. 10·7

$$\therefore \frac{\text{Area of } A'B'C'D'E'..}{\text{Area of } ABCDE....} = \frac{[l(A'B')]^2}{[l(AB)]^2} \quad \ldots\ (i)$$

But as shown in § 10·13, we have

$$\frac{l(A'B')}{l(AB)} = \frac{l(OA')}{l(OA)} = \frac{l(ON')}{l(ON)} \quad \ldots\ (ii)$$

By (i) and (ii)

$$\frac{\text{Area of } A'B'C'D'E}{\text{Area of } ABCDE} = \frac{[l(ON')]^2}{[l(ON)]^2} \text{ as required.}$$

Theorem 10·5 : The volumes of similar tetrahedra (triangular pyramids) are in the same ratios as the cubes of their corresponding edges.

Proof : Let V and V′ be the volumes of two similar tetrahedra O—ABC and O′—A′B′C′ with corresponding pairs of edges seg AB and seg A′B′, seg BC and seg B′C′, seg CA and seg C′A′. Let a, b, c, a', b', c' be the lengths of the edges seg BC, seg CA, seg AB, seg B′C′, seg C′A′ and seg A′B′ respectively. Let seg ON and seg O′N′ be the altitudes of these pyramids. Let h and h' be the lengths of ON and O′N′.

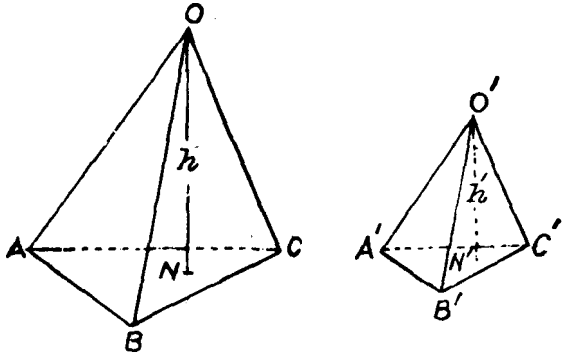


Fig. 10·8

To prove : $\frac{V'}{V} = \frac{(a')^3}{a^3} = \frac{(b')^3}{b^3} = \frac{(c')^3}{c^3}$

Proof : $\dfrac{V'}{V} = \dfrac{(\text{Area of base } A'B'C') \cdot h'}{(\text{Area of base } ABC) \cdot h}$ ... . (i)

But $\dfrac{\text{Area of base } A'B'C}{\text{Area of base } ABC} = \dfrac{[l(A'B')]^2}{[l(AB)]^2} = \dfrac{(c')^2}{c^2}$ .... (ii)

Also $\dfrac{h'}{h} = \dfrac{l(O'N')}{l(ON)} = \dfrac{l(A'B')}{l(AB)} = \dfrac{c'}{c}$ ... . (iii)

From (i), (ii) and (iii), we get

$$\frac{V'}{V} = \frac{(c'^2)}{c^2} \cdot \frac{c'}{c} = \frac{(c')^3}{c^3}$$

Similarly, $\dfrac{V'}{V} = \dfrac{(a')^3}{a^3} = \dfrac{(b')^3}{b^3}$. $\therefore$ $\dfrac{V'}{V} = \dfrac{(a')^3}{a^3} = \dfrac{(b')^3}{b^3} = \dfrac{(c')^3}{c^3}$.

§ 10·14. Examples :

Ex. 1. A right pyramid of height 12 cms. stands on a square base whose side is 10 cms. Find (i) the slant height, (ii) the slant surface and (iii) the volume of pyramid.

Ans. Let O – ABCD be the regular pyramid where ABCD is a square. Let OM be a perpendicular to ABCD. Let MN be perpendicular to BC as shown in fig. 10·9. Then triangle OMN is a right angled triangle with $l(OM) = 12$ cms. and $l(MN) = 5$ cms.

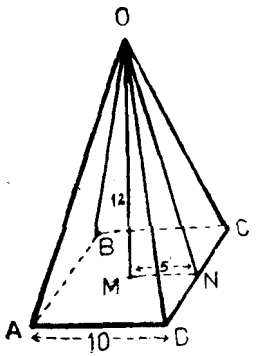


Fig. 10·9

$\therefore$ $[l(ON)]^2 = [l(OM)]^2 + [l(MN)]^2$
$= 144 + 25 = 169$

$\therefore$ $l(ON) = 13$.

$\therefore$ *Slant height = 13 cms.*

Slant surface $= \frac{1}{2}$ (Perimeter of base) $\times$ (slant height)

$= \frac{1}{2}(10 + 10 + 10 + 10)(13) = 260$ sq. cms.

Volume $= \frac{1}{3}$ (area of base) (height) $= \frac{1}{3}(100)(12) = 400$ cu.cms.

Ex. 2. The ends of a frustrum of a pyramid are equilateral triangles, the lengths of the sides being 6′ and 7′ respectively and the height of the frustrum is 9′, Find its volume.

Ans. Since the ends of the frustrum are equilateral triangles, their areas are

$\dfrac{\sqrt{3}(6)^2}{4}$ and $\dfrac{\sqrt{3}(7)^2}{4}$, i. e. $9\sqrt{3}$ sq. ft. and $\dfrac{49}{4}\sqrt{3}$ sq. ft.

$\therefore$ Volume of frustrum $= \frac{1}{3}$ (height) $(S_1 + S_2 + \sqrt{S_1 S_2})$

where S_1 and S_2 are the areas of its ends.

$\therefore$ Required volume

$$= \frac{1}{3} (9) \left(9\sqrt{3} + \frac{49\sqrt{3}}{4} + \sqrt{9\sqrt{3} \cdot \frac{49\sqrt{3}}{4}} \right)$$

$$= 3 \left(9\sqrt{3} + \frac{49\sqrt{3}}{4} + \frac{3(7\sqrt{3})}{2} \right)$$

$$= 3\sqrt{3} \left(9 + \frac{49}{4} + \frac{21}{2} \right)$$

$$= 3\sqrt{3} \left(\frac{127}{4} \right) = \frac{381\sqrt{3}}{4} \text{ cu. ft.}$$

Exercise 10·1 :

(1) Find the area of the base of a regular square pyramid whose lateral faces are equilateral triangles and whose altitude is 4 cms.

(2) A solid is bounded by four equilateral triangles, the side of each triangle being 12 in. Find the volume.

(3) The faces of a pyramid on a square base are equilateral triangles, the side of the base being 20 mts. Find the volume.

(4) A pyramid has for its base an equilateral triangle of which each side is 2 m. and its slant edge is 6 m. Find its volume.

(5) The base of a pyramid is a square and its faces equilateral triangles. Prove that its volume is $\frac{a^3}{6}\sqrt{2}$, where a is the side of the base.

(6) The sides of a rectangular tank measure 30 ft. and 20 ft. at the top and 12 ft. and 8 ft. at the bottom. Find the volume of the tank if the depth is 8 ft.

(7) The ends of a frustum of a pyramid are squares, the lengths of the sides being 20 ft. and 30 ft. respectively. The length of the straight line whic h joins the middle points of the two ends is 13 ft. Find the volume.

(8) The height of a frustum of a pyramid is 12·5 in., its ends are octagons whose sides are 4 and 2 in. respectively : find the volume of the frustum.

§ 10·2. Cone :

Definition : A solid generated by a straight line which moves with one end fixed and the other end moving along a plane curve is called a cone.

Note: A cone may also be defined as the limit of a pyramid the number of sides of whose base $\to \infty$ and the length of each side of the base $\to 0$.

Notations: The fixed point is called the *vertex* of the cone, the moving straight line is called the *generator* and the guiding curve is called the *base*. The line joining the vertex to the centre of the base is called the *axis* of the cone.

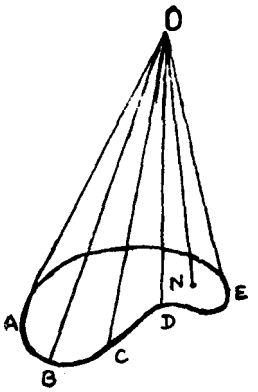


Fig. 10·10

A cone is shown in fig. 10·10. O is the vertex, OA is the generator and ABCDEF .... is the base. ON is the axis of the cone.

If the base of a cone is a circle as in fig. 10·11, the cone is called a *circular cone*. If the axis of the cone is perpendicular to the base as in fig. 10·12, the cone is called a *right cone*. If the axis of a circular cone is perpendicular to the base as in fig. 10·13, the cone is called a *right circular cone*.

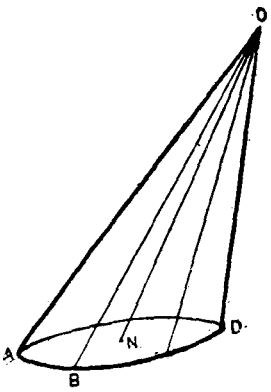


Fig. 10·11

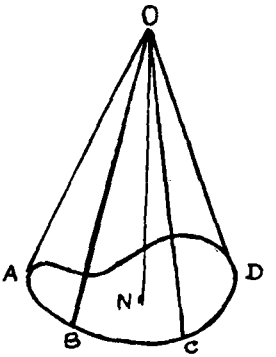


Fig. 10·12

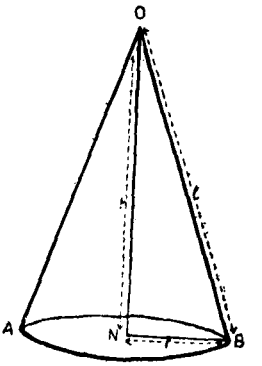


Fig. 10·13

§ 10·21. Volume and lateral area of a right circular cone :

If we inscribe a regular pyramid of n faces in a right circular cone, then, as $n \to \infty$, the inscribed pyramid $\to$ the right circular cone. With the help of this concept of limit, we can find the

volume of the right circular cone by taking the limit as $n \to \infty$, of the volume of the inscribed regular pyramid of n faces.

Theorem 10·6: The volume of a right circular cone is equal to one-third the product of the area of its base and altitude.

Proof: Let O – ABCDEF .... be the given right circular cone. Let O – ABCDEF .... be a regular pyramid of n faces inscribed in this right circular cone with vertex at O and with its base along the circular base.

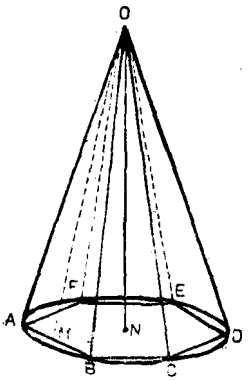


Fig. 10·14

$\therefore$ Height of the pyramid is the same as the height of the cone

As $n \to \infty$, the inscribed regular pyramid $\to$ the right circular cone.

$\therefore$ Volume of the right circular cone

$= \lim_{n \to \infty}$ (volume of the inscribed regular pyramid)

$= \lim_{n \to \infty} \frac{1}{3}$ (area of the base of pyramid)·(height of the pyramid)

$= \frac{1}{3}$ (area of the circular base of the cone)·(height of the cone)

$= \frac{1}{3}$ (area of the base) · (altitude).

Corollary: If r is the radius of the base and h the altitude of a right circular cone, then its volume is $\frac{1}{3} \pi r^2 h$.

Proof: By theorem 10·6,

Volume of the right circular cone

$= \frac{1}{3}$ area of the base $\times$ altitude

$= \frac{1}{3}$ · (area of the circle with radius r)·h

$= \frac{1}{3} (\pi r^2) \cdot (h) = \frac{1}{3} \pi r^2 h.$

Note : The volume of a circular cone of height h is equal to that of a right circular cone of height h with the same circle as base.

Theorem 10·7 : The lateral area of a right circular cone is $\frac{1}{2}$ (circumference of the base) · (slant height).

Proof :

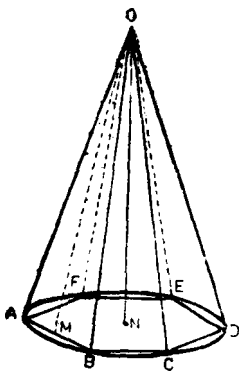


Fig. 10·15

Let O – ABCDE .... be the given right circular cone. Let O – ABCDEF .... be a regular pyramid of n faces inscribed in this right circular cone with vertex at O and with its base along the circular base.

$\therefore$ As $n \to \infty$, the inscribed regular pyramid $\to$ the right circular cone and the perimeter of the regular polygonal base $\to$ the circumference of the circular base.

Let OM be the altitude of the triangle OAB. $\therefore$ OM is the slant height of the pyramid. As $n \to \infty$, this slant height of the regular pyramid $\to$ the slant height of the right circular cone.

Lateral area of the right circular cone

$= \lim_{n \to \infty}$ (lateral area of the inscribed regular pyramid).

$= \lim_{n \to \infty} \frac{1}{2}$ (perimeter of the polygonal base) · (slant height of the pyramid).

$= \frac{1}{2}$ (circumference of the circular base) · (slant height of the cone).

Corollary (1) : The lateral area of a right circular cone of slant height l and the radius of base r is πrl.

Proof : By theorem 10·7,

the lateral area of the right circular cone

$= \frac{1}{2} \cdot$ (circumference of the base) · (slant height)

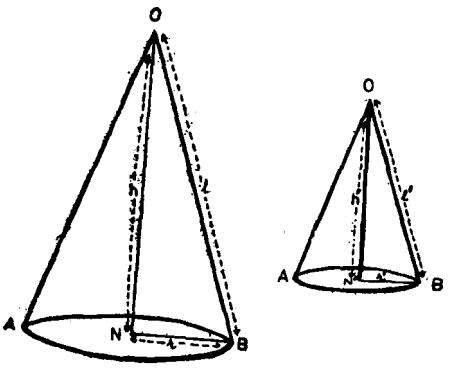
$= \frac{1}{2} \cdot 2\pi r \cdot l$

$= \pi rl$ as required.

Corollary (2): The total surface area of a right circular cone whose slant height is l and the radius of whose base is r is $\pi rl + \pi r^2$.

Proof: Total surface area of a right circular cone
= its lateral area + area of its base
$= \pi rl + \pi r^2$ as required.

Theorem 10·8:



Fig, 10·16

The volumes of similar right circular cones are proportional to the cubes of the corresponding dimensions.

Proof: Let V, V′ be the volumes, h, h' the altitudes, r, r' be the radii of the bases and l, l' be the lengths OB and O′B′ respectively of the right circular cones in fig. 10·16.

To prove:

$$\frac{V'}{V} = \frac{(l')^3}{l^3} = \frac{(r')^3}{r^3} = \frac{(h')^3}{h^3}$$

Proof: By corollary on theorem 6·6,

$$\frac{V'}{V} = \frac{\frac{1}{3}\pi (r')^2 h'}{\frac{1}{3}\pi \cdot r^2 \cdot h} = \frac{(r')^2}{r^2} \cdot \frac{h'}{h} \cdot \quad \text{But } \frac{r'}{r} = \frac{h'}{h} = \frac{l'}{l}$$

$$\therefore \quad \frac{V'}{V} = \frac{(r')^2}{r^2} \cdot \frac{h'}{h} = \frac{(h')^2}{h} \cdot \frac{h'}{h} = \frac{(h')^3}{h^3}$$

$$\therefore \quad \frac{V'}{V} = \frac{(h')^3}{h^3} = \frac{(r')^3}{r^3} = \frac{(l')^3}{l^3}. \quad \text{Hence the theorem.}$$

§ 10·22. Section of a right circular cone:

For the cone O – ABC given in fig. 10·17, the base ABC is a circle with centre N and radius r. Further, since it is a right circular cone, ON is the altitude. Let its length be h.

Let $l(OA) = l(OB) = \ldots = l$.

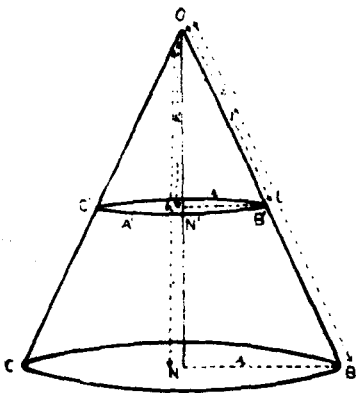


Fig. 10·17

Let the section of this cone by a plane A'B'C' drawn parallel to the circular base ABC. Now this section is parallel and similar to the base. Hence, the section A'B'C' is also a circle, i. e. *the section of a right circular cone by a plane parallel to its base is a circle.*

Let the altitude ON meet this section at N'. ∴ N' is the centre of the circular section. Let

$$l(OA') = l(OB') = l',$$

$$l(ON') = h',$$

and radius of the section i. e. $l(N'B') = r'$.

Now since seg N'B' ∥ seg NB, triangle ON'B' = triangle ONB.

$$\therefore \quad \frac{l(ON')}{l(ON)} = \frac{l(OB')}{l(OB)} = \frac{l(N'B')}{l(NB)}, \text{ i. e. } \frac{h'}{h} = \frac{l'}{l} = \frac{r'}{r}.$$

i. e. the altitude and other elements of a right circular cone are divided proportionally by a plane parallel to its base.

Theorem 10·9 : The area of the section of a right circular cone cut by a plane parallel to its base is to the area of the cone as the square of its distance from the vertex to the square of the altitude of the cone.

Proof : Let the circle P'Q'R' be the section of the right circular cone O – PQR, by a plane parallel to the circular base PQR. Let r and r' be the radii, N and N' be the centres of the circles PQR and P'Q'R'. Since the cone is a right circular cone, $l(ON)$ and $l(ON')$ are the distances of the circles PQR and P'Q'R' from the vertex O. Let $l(ON') = h'$ and $l(ON) = h$. ∴ h is the altitude. Let A and A' be the areas of the circles PQR and P'Q'R' respectively.

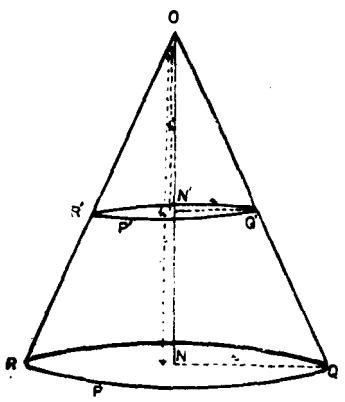


Fig. 10·18

To prove : $\frac{A'}{A} = \frac{h'^2}{h^2}$

Proof : $\frac{A'}{A} = \frac{\pi r'^2}{\pi r^2} = \frac{r'^2}{r^2}$ .... (i)

but the triangle ON'Q' = triangle ONQ

$\therefore \frac{l(N'Q')}{l(NQ)} = \frac{l(ON')}{l(ON)}$. $\therefore \frac{r'}{r} = \frac{h'}{h}$ .... (ii)

From (i) and (ii) $\frac{A'}{A} = \frac{h'^2}{h^2}$.

§ 10·23. Examples :

Ex. 1. Find the volume and the lateral area of a right circular cone whose height is 4 ft. and the radius of whose base is 3 ft.

Ans. Here $h = 4'$ and $r = 3'$

$\therefore$ Slant height $l = \sqrt{4^2 + 3^2} = 5'$

Now volume of cone $= \frac{1}{3} \pi r^2 \cdot h$

$= \frac{1}{3} \pi (9)(4) = 12\pi$ cu. ft.

and lateral area $= \pi r l = \pi (3)(5) = 15\pi$ sq. ft.

Ex. 2. A conical tent of canvas is to be built to accommodate 72 people under the following conditions. The amount of canvas available is 550 sq. m. and each person must have 2 sq. m. space on the ground. Find the height of the tent so built and also the amount of air available for breathing to every person. $\left(\text{Take } \pi = \frac{22}{7} \right)$.

Ans. Let the required cone have height h mts., radius of base r mts. and slant height l mts.

We are given that each purson must have a floor space of 2 mts. and there are 72 persons.

$\therefore$ area of base $= 2 \times 72 = 154$ sq. m.

$\therefore \pi r^2 = 154 \quad \therefore \frac{22}{7} \cdot r^2 = 154. \quad \therefore r^2 = \frac{154 \times 7}{22} = 49.$

$\therefore r = 7$ metres.

Also the amount of canvas used = 550 sq. m.

$\therefore$ lateral area = 550 sq. m.

$\therefore \pi \cdot (7) l = 550, \quad \therefore \frac{22}{7} \cdot 7 \cdot l = 550 \quad \therefore l = \frac{550}{22} = 25$ m.

Now $r^2 + h^2 = l^2 \quad \therefore h^2 = l^2 - r^2 = (25)^2 - (7)^2 = (24)^2$

$\therefore$ $h = 24$ metres is the height of the cone.

Now amount of air available

$$= \text{volume of the cone} = \tfrac{1}{3}\pi r^2 \cdot h$$

$$= \frac{1}{3} \cdot \frac{22}{7}(49) \cdot (24) = 1232 \text{ cu. mts.}$$

This amount of air is available for 72 people.

$\therefore$ Each gets $\frac{1232}{72} = 17{\cdot}11$ cu. mts. approximately.

Ex. 3. The frustrum of a cone, made of copper, has height 6 cms. and the radii of bases are 3 cms. and 2 cms. It is melted away and drawn into a solid rod of length 38 cms. and of uniform cross section. Find the diameter of the rod if there is no loss of metal during the transformation.

Ans. If a frustrum of a cone has height h and radii of base r_1 and r_2, its volume is $V = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)$

$\therefore$ Volume of the given frustrum $= \frac{\pi}{3}(6)(3^2 + 3 \times 2 + 2^2)$

$$= 2\pi(19) = 38\pi \text{ cu. cms.}$$

This volume of metal is turned into the rod. Let the radius of the cross section of the rod of length 38 cms. be r.

$\therefore$ Volume of the rod $= \pi \cdot r^2 (38)$ cu. cms.

$\therefore$ $38\pi r^2 = 38\pi$ $\quad\therefore\quad r^2 = 1$ $\quad\therefore\quad r = 1$ cm.

Hence the diameter of cross section $= 2$ cms.

Ex. 4. The depth of a pail, in the form of the frustrum of a cone is 10″ and the diameters of its mouth and base are 12″ and 8″ respectively. If a conical reservoir of depth 19″ and diameter of base 40″ is to be filled using this pail, how many times will the pail have to be emptied in it?

Ans. Since depth of pail $= 10''$ and the radii of the mouth and bottom are 6″ and 4″,

Volume of the pail $= \frac{1}{3}\pi(10)(6^2 + 4^2 + 6 \times 4)$

$$= \frac{10\pi}{3}(76) = \frac{760\pi}{3} \text{ cu. in.}$$

Now volume of the cone with height 19″ and radius of base 20″

$$= \tfrac{1}{3}\pi(20)^2 \cdot (19) = \frac{7600\pi}{3} \text{ cu. in.}$$

$\therefore$ Number of times the pail must be emptied into the tank

$$= \frac{7600\pi}{3} \div \frac{760\pi}{3} = 10.$$

Exercise 10·2 :

(1) Find the volume of a right circular cone, if the radius of its base is 3 m. and its height is 10 m.

(2) The slant side of the frustum of a right circular cone is 5 ft., and the radii of the ends are 7 ft. and 10 ft.; find the volume.

(3) A conical oil can is 20 inches high and has a base of diameter 9 inches. It is cut off in the middle to form a bucket. Compare the volumes of the portion removed and of the bucket.

(4) A semi-circular sheet of metal of diameter 56 cms. is bent into an open conical cup. Find the greatest radius and depth of the cup.

(5) A conical tent is required to accommodate 157 persons; each person must have 2 sq. metres of space on the ground and 15 cu. metres of the air to breathe. Find the height of the tent and also calculate the slant height. ($\pi = 3{\cdot}14$).

(6) A right angled triangle, whose remaining angles are 60° and 30°, revolves completely about its hypotenuse, which is 12 in. long. Find the volume of the solid thus described.

(7) The frustum of a cone, made of copper, has its height 12 cms. and the radii of its bases are 6 cms. and 4 cms. It is melted away and drawn into a solid rod of length 76 cms. and of uniform cross-section. If there be no loss of the metal during the process of transformation, find the diameter of cross-section of the rod.

(8) Two buckets have the same depth of 84 cms. each. One is cylindrical with 75 cms. as its diameter while the other is a conical frustum, the diameters of its ends being 85 cms. and 65 cms. respectively. Which will yield more water and by how much ?

(9) A metal piston is in the shape of the frustum of a cone, the top and bottom diameters being 4 cms. and 6 cms. respectively. It has a height of 7 cms. and one cubic centimetre of the metal weighs $\left(1\frac{1}{11}\right)$ gms. Find the weight of the piston.

(10) The depth of a pail in the form of a frustum of a cone is 7″, its diameter at the mouth is 12″ and its diameter at the bottom is 8″. Find how often it can be filled from a tank containing 8360 cu. inches of water.

(11) Find the diameter of the cross-section of a metal rod 36 ft. long that can be made out of the largest possible right circular cone,

cut out from a solid metal cube whose edge is 3 ft. Assume that there is no loss of the metal during the whole process of conversion.

(12) A right circular solid cone, made of copper, has the radius of its base 4 ft. Its upper part is cut off by a plane at right angles to its axis. The truncated portion of the cone is then melted away and drawn into a fine wire of uniform cross-section and of length 7,000 yds. Find the diameter of the cross-section of the wire.

(13) A bucket is in the shape a conical frustum (height = 9 in., diameters of top and bottom surfaces = 10 in. and $7\frac{1}{2}$ in. respectively). Find how much lower the water will stand in a well, whose diameter is 5 ft., after the bucket has been filled twenty-four times.

§ 10·3. Sphere :

A *sphere* is the locus of a point which moves in space such that it remains at a constant distance from a fixed point. The fixed point is called the *centre* and the constant distance is called the *radius* of the sphere.

If we take a circle whose diameter is seg AB as shown in fig. 10·19, and rotate it completely about seg AB then we get a sphere whose diameser is seg AB and whose centre is this mid-point of seg AB.

Theorem 10·10 : The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Proof : Consider the circle $x^2 + y^2 = r^2$. Then if the shaded area of the semicircle in fig. 10·19 is revolved around x axis, we shall get the sphere of radius r. Obviously, the points A and B where the circle meets the x axis are A $(r, 0)$ and B $(-r, 0)$. ∴ The required volume of the sphere

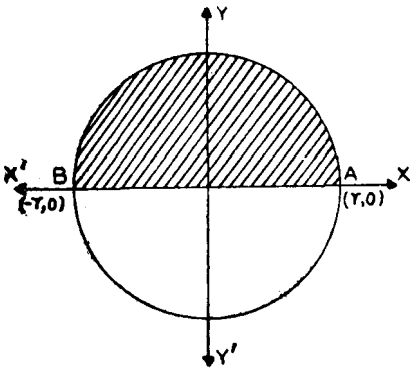


Fig. 10·19

$$= \int_{-r}^{r} \pi y^2 \, dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2) \, dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right] = \frac{4}{3} \pi r^3.$$

Theorem 10·11: The volumes of two spheres are to each other as the cubes of their radii.

Proof: Let V and V′ be the volumes of two spheres, whose radii are r and r' respectively.

$$\therefore V = \frac{4}{3}\pi r^3 \text{ and } V' = \frac{4}{3}\pi (r')^3$$

$$\therefore \frac{V}{V'} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi (r')^3} = \frac{r^3}{(r')^3}.$$ Hence the result.

§ 10·31. Section of a sphre by a plane:

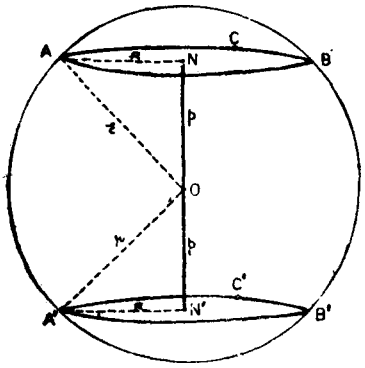


Fig. 10·20

Let O be the centre and r be the radius of a sphere. Let ABC be the section of this sphere by a plane. Let N be the foot of the perpendicular drawn from the centre O to the section.

Let $l(ON) = p$. $\therefore$ seg ON is fixed. $l(OA) = r$ is also fixed. Hence length of seg AN is fixed, i. e. constant, say R where $R = \sqrt{r^2 - p^2}$. Similarly, $l(BN) = R$, $l(CN) = R$, etc.

$\therefore$ The section ABC is a circle whose radius is R. Hence *every plane section of a sphere is a circle.* If such a circle passes through the centre of the sphere, it is called a great circle. Two great circles of a sphere bisect each other.

Further if we have another section A′B′C′ of the same sphere such that the length of the perpendicular ON′ $= p$, then as shown earlier $l(A'N')$ is also $\sqrt{r^2 - p^2}$ i. e. R.

Hence *the circles cut by the planes equidistant from the centre of a sphere are congruent and conversely if the circles cut by the planes on a sphere are congruent, then they will be equidistant from the centre.*

We assume without proof the following result about the surface area of a sphere. "The surface area of a sphere of radius r is $4\pi r^2$." Hence the areas of two spheres are to each other as the squares of their radii or diameters.

§ 10·32. Spherical Segment and Spherical Cap:

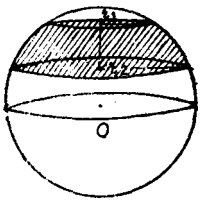


Fig. 10·21

The solid bouuded by a sphere and two parallel planes that intersect it is called a *spherical segment* (Fig. 10·21). If one of the parallel planes is a tangent to the sphere then the segment is known as a *spherical cap.* (Fig. 10·22).

The sections (i. e. circles) made by the two parallel planes are called the *bases* of the segment. The altitude of the spherical segement is the distance between the parallel planes.

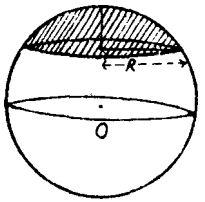


Fig. 10·22

Consider a sphere of radius r. If r_1 and r_2 are the radii of the bases of a spherical segment of height h and R is the radius of the base of a spherical cap of height H, then the following results will be assumed without proof.

(1) Area of the curved surface of the segment $= 2\pi rh$

(2) Area of the total surface of the segment

$$= 2\pi rh + \pi r_1^2 + \pi r_2^2.$$

(3) Volume of the segment $= \frac{\pi h}{6}(3r_1^2 + 3r_2^2 + h^2)$.

A spherical cap can be considered as a spherical segment the radii of whose bases are 0 and R as shown in fig. 10·22.

$\therefore$ Volume of spherical cap of height H

$$\therefore \quad \frac{\pi H}{6}(3R^2 + 0 + h^2) = \frac{\pi H}{6}(3R^2 + H^2)$$

Similarly, the area of the curved surface is $2\pi rH$ and the total surface is $2\pi rH + \pi R^2$.

§ 10·33. Examples :

Ex. 1. Find the volume and the surface area of a sphere whose radius = 3·5 ft. $\left(\text{Take } \pi = \frac{22}{7}\right)$.

Ans. Volume $V = \frac{4}{3}\pi r^3 = \frac{4}{3} \cdot \frac{22}{7}\left(\frac{7}{2}\right)^3 = \frac{4}{3} \cdot \frac{22}{7} \cdot \frac{7^3}{8}$

$= 179{\cdot}66$ cu. ft.

Surface area $= 4\pi r^2 = 4 \cdot \frac{22}{7} \cdot \left(\frac{7}{2}\right)^2 = 154$ sq. ft.

Ex. 2. How many solid spheres, each 12 cm. in diameter, could be moulded from a solid metal cylinder of height 10 cms. and diameter 24 cms. ?

Ans. Since the cylinder has radius 12 cms. and height 10 cms., its volume is $\pi \cdot (12)^2 \cdot (10) = 1440\pi$ cu. cms. Also the volume of one spherical ball of radius 6 cms. is $\frac{4}{3} \cdot \pi \cdot (6)^3 = 288\pi$ cu. cms.

$\therefore$ Number of balls that can be moulded from the cylinder

$$= \frac{1440\pi}{288\pi} = 5.$$

Ex. 3. A hemispherical tank has a diameter of 12′. It is to be filled from a pipe 4″ in diameter discharging water at the rate of 36 ft. per minute. Find how it takes to fill the tank.

Ans. Volume of the hemispherical tank with radius 6′ is $\frac{1}{2} \cdot \frac{4}{3} \cdot \pi (6)^3 = 144\pi$ cubic ft.

If one minute, the volume of water poured into the tank is equal to the volume of a solid cylinder of height 36 ft. and radius of base $2'' = \left(\frac{1}{6}\right)'$.

$\therefore$ Volume of water poured per min. $= \pi\left(\frac{1}{6}\right)^2 \cdot (36) = \pi$ cub ft.

$\therefore$ Time to fill the tank $= \frac{144\pi}{\pi} = 144$ min. $= 2$ hrs. and 24 min.

Ex. 4. Find the volume of the segment of a sphere of radius 5 ft. such that the thickness of the segment is 1 foot and the radius of its large end is 4 ft.

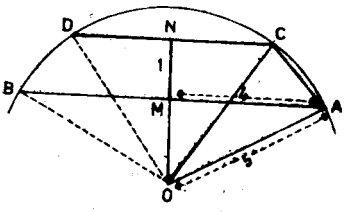


Fig. 10·23

Ans. Here ABCD is the zone. (Fig. 10·23)

∴ If O is the centre of the sphere, $l(OA) = l(OC) = 5'$, $l(AM) = 4'$ and $l(MN) = 1'$.

Now $[l(OM)]^2 = [l(OA)]^2 - [l(AM)]^2 = 25 - 16 = 9$.

∴ $l(OM) = 3'$

∴ $l(ON) = 3 + 1 = 4'$

Now $[l(NC)]^2 = [l(OC)]^2 - [l(ON)]^2$

$= 25 - 16 = 9$ ∴ $l(NC) = 3'$

∴ The radii of the zone are 3′ and 4′.

∴ Volume of zone $= \frac{\pi}{6} \cdot (1)\,[3\,(9) + 3\,(16) + 1]$

$= \frac{\pi}{6} \cdot (27 + 48 + 1)$

$= \frac{\pi}{6} \cdot (76) = \frac{38\pi}{3}$ cubic ft.

Exercise 10·3 :

(1) A sphere is 36 cms. in diameter. Find its volume and surface area

(2) A vessel is a hollow cylinder fitted with a hemispherical bottom of the same diameter. The depth of the cylinder is 2·5 metres and its diameter 2·1 metres. Calculate, in litres, the volume of water it can hold. (1 litre = 1,000 cc.)

(3) Indicate why you consider the following statement wrong : If the area of a country on a globe of 10 cms. diameter is shown as 8 sq. cms., then the area of the same country, on a globe of double the diameter, will be 16 sq. cms.

(4) A cylindrical vessel has a base of diameter 10 inches. It is completely filled by emptying into it the contents of a hemispherical bowl also of diameter 10 inches. Find the height of the cylinder.

(5) How many solid spheres, each 6 cms. in diameter, could be moulded from a metal cylinder whose height is 45 cms. and diameter 4 cms.?

(6) How many spherical balls of 2·5 mm. diameter each can be obtained by melting a semi-circular disc of 7 cms. diameter and 8 mm. thickness?

(7) A hemispherical dome, with an inside diameter of 8 ft., is to be whitewashed both on the outside and the inside. If the thickness of the wall be 9 inches, find the cost at Rs. 14 per 100 sq. ft.

(8) Find the volume of metal in a spherical shell with an external diameter of 8 cms. and thickness 1 cm. What would be the weight of 21 such equal shells, if the metal weighs 5 gms. per cu. cm. ?

(9) A solid sphere of radius 4 inches is blown into a hollow sphere of uniform thickness. If the radius of the external surface be 5 inches, find the thickness of the hollow sphere. (Take $\sqrt[3]{61}=3{\cdot}94$).

(10) A hollow sphere 1 inch thick has an outside diameter of 10 inches. The liquid filling it is allowed to pass into a cylindrical vessel with a base of radius 9 inches. How high will it rise in the latter ?

(11) A hemispherical tank has a diameter, of 12 feet. It is to be filled from a pipe 2 inches in diameter, discharging fluid into it at the rate of 18 feet per minute. How long does it take to fill the tank ?

(12) The radius of the cap of a sphere is 24 cms. and the height of the cap is 12 cms. Find its volume.

(13) The radius of the base of a metallic spherical cap is 10 cms. and its height is 8 cms. If the cap is hammered out into a circular plate 80 cms. in diameter, find the thickness of the plate.

(14) Find the volume of the segment of a sphere of radius 10 cms. if the thickness of the segment is 2 cms. and the radius of its large end is 8 cms.

(15) A heavy porous spherical ball capable of absorbing water to a quantity of one-fourth its volume is dropped into a vessel in the form of a right circular cylinder, in which water stands at a height of 10 cms. and is completely immersed. Find the distance by which the level of water will change when it is given that the diameter of the ball is 6 cms. and the radius of the circular base of the cylinder is 4 cms.

(16) Four heavy porous spherical balls are dropped one by one within a right circular cylinder three-fourths of which is filled with water. The height of the cylinder is 15 cms. and the diameter of the cross-section of the base is 16 cms. If the diameters of the balls are 2 cms., 4 cms., 6 cms. and 8 cms. respectively and if each ball has the capacity of absorbing three-eights of its own volume of water, find, correct to three places of decimal, the change in the height of the water level when the balls have been all dropped.

§ 10·4. Cavalieri's Principle :

This is a very useful principle for calculating the volumes of irregular solids. The statement is as follows :

'If in two solids of equal heights, the sections made by planes parallel to and at the same distance from their respective plane bases are always equal, the volumes of the solids are equal.'

This is illustrated in fig. 10·24, where two solids of different shape and same height are shown. When 3 planes parallel to their bases and at the same distance from the bases cut these two solids as shown in fig. 10·24, the sections of the two solids which are polygons in the first case and circles in the other, have the same area. If this happens for all such parallel planes, then by Cavalieri's principle, the solids have the same volume.

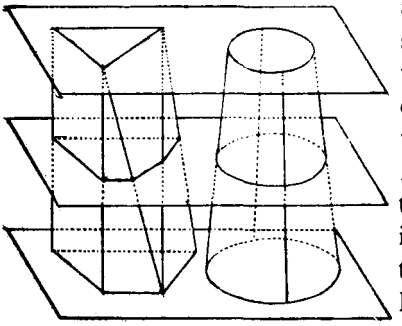


Fig. 10·24

We shall not prove this result but explain it as follows:—

A solid can be considered as being made up of a large number of discs of very small thickness, placed one obove the other. If the number of such discs becomes infinitely large and the thickness of each disc tends to zero, then each disc becomes a section by a plane.

Since the areas of all corresponding sections of the two solids are equal when such infinite number of areas are summed up, the resultant volumes are also equal.

As an example, consider two right prisms, one with rectangular base and the other with a square base, having the same

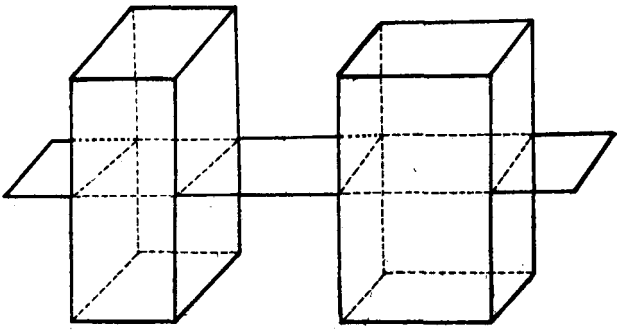


Fig. 10·25

height, such that the side of the square is 6 cms. and the sides of the rectangle are 9 and 4 cms. Then each cross section of the two figures by a plane parallel to the base is of area 36 sq. cms. and hence the stwo solids have equal volume by Cavalieri' principle.

CHAPTER 11

Frequency Distributions

§ 11·1. Introduction:

Broadly speaking, *Statistics* deals with the collection, classification, tabulation, analysis and interpretation of data. It is helpful in deciding what data is needed, how the data should be collected, classified and analysed and how the results obtained from the analysis can be generalized to determine possible future trends. Thus, statistical analysis of past experience is helpful in making decisions for the future and therefore is useful in any branch which involves decision-making based on past experience. Statistics is particularly useful to the managers, business executives, administrators, government officials in any field where making such decisions is essential. Let us examine with the help of an example how this is done in the case of a company manufacturing electric bulbs by two different methods A and B. The manager of the company is interested in finding whether the bulbs manufactured by method A, on an average, last longer than those manufactured by method B, so that he may produce more bulbs using one method than the other.

Suppose that this company manufactured 90,000 bulbs in a year by each method. Naturally, in order to compare the two methods, as regards the life of the bulbs, it is necessary to find the averages, say m_1 and m_2, of the life of each set of bulbs manufactured by the two methods A and B. Before we can find m_1, the average life in hours, of the set of bulbs manufactured by method A, we have to put each one of these bulbs on test to find the life, in hours, of each one of these 90,000 bulbs. Suppose this experiment gives the life of bulbs as 267, 235, 283, 297, ...hours. This list contains 90,000 numbers which represent the life of each bulb manufactured by method A. Similarly, 90 000 figures representing the life of the bulbs manufactured by method B can also be found. We can then find averages of these two sets of numbers and can finally find m_1 and m_2, and then the manager can draw his own conclusion depending on whether $m_1 < m_2$,

$m_1 = m_2$ or $m_1 > m_2$ and take necessary decision to improve the same and increase the profits of the company. He wants such a conclusion in order to increase the sales and profits but before he can find m_1 and m_2, he has to lose all 180,000 bulbs manufactured by the company. Naturally he has nothing now to sell. Not only that, he incurs such heavy losses that he may have to close down the company. Naturally a manager cannot take decisions in this way.

Now let us see how he may think of solving this problem. He may select a sample of say 90 bulbs from each of the two sets, put them on test, get the numbers of representing the lives of these bulbs and then get two averages corresponding to the two samples. Naturally, these averages cannot be expected to be the same as m_1 and m_2, because the average life of 90 bulbs cannot be exactly the same as that of 90,000 bulbs, but even then the averages of two groups of 90 bulbs are practically more useful than finding the exact averages for the entire groups of 90,000 bulbs. With the help of the averages obtained from the samples, we may generalise for the complete groups, and the study of Statistics will help us to know the best way to do this. Statistics primarily helps us in making decisions based on samples.

With the help of this example, let us discuss some of the terms used in Statistics. Here we have a statistical enquiry, the object of which is to study the life of the bulbs manufactured by the two methods. Let us consider only one sample of 90 bulbs selected from the set of 90,000 bulbs manufactured by method A. Here this set of 90,000 bulbs is the *population.* Each one of these bulbs, is an *individual of the population* and the number of individuals in the population, viz. 90,000, is called the *size* of the population or population size and m_1 is called the *population average.*

The small group of 90 bulbs selected from the popultion is called a *sample* from the population. Each one of these 90 bulbs is an *individual of the sample*, and the *sample size* is 90. The average obtained from the sample is known as the *sample average.*

The characteristic that we study here for each individual (i. e. a bulb) is its life in hours. Here we can actually find the number of hours for which the bulb may last, i. e. the characteristic can

be numerically specified by numbers such as 267, 235,....etc. This characteristic is said to be *quantitative*. Sometimes instead of finding the number of hours, we may be interested in finding whether the bulb is defective or not defective. In such a case, we cannot write this characteristic in numbers, we can only classify the bulbs in groups "either defective or not defective". We cannot say that one bulb is more defective than the other. Such a characteristic of an individual is said to be a *qualitative* characteristic.

A quantitative characteristic is called a *variate*, whereas a qualitative characteristic is called an *attribute*. We now define all these terms in general.

§ 11·11. Definitions:

(1) *Population* is any well-defined set of all objects about which a statistical enquiry is being made.

(2) Each object belonging to the population is called an *individual of the population.* The total number of individuals in a population is called the size of the population. The number of individuals in a population may be finite or infinite.

(3) A *sample* is any finite set of objects drawn from the population, with an aim to make observations from them.

(4) Each object belonging to a sample is called an *individual of the sample.* The total number of individuals in a sample is called the *sample size.* Sample size is always finite.

The characteristic of an individual under study is either quantitative or qualitative.

(5) An *attribute* is the qualitative characteristic of an individual which cannot be expressed numerically but according to which an individual may be classified as belonging to one out of two or more disjoint groups.

(6) A *variate* is the quantitative characteristic of an individual which can be expressed numerically and for which each individual can be given a definite numerical measure. The different values which variate takes are called the *values of a variate.*

Let us now examine these concepts with the help of some examples.

Ex. 1. A factory manager orders, every week, a large number of bolts of length 1". He can use these bolts only if the length of the bolt received is between 0·9" and 1·1". He agrees to accept the lot from the seller if this condition is satisfied. Based on this, he has to decide whether to accept or reject the lot.

Here the statistical enquiry is relating to the length of the bolts received by the factory manager and the object of the study is to decide whether to accept or reject the lot of bolts received by him. Each bolt is an individual. The characteristic of the individual is (i) the length of bolt in inches, or (ii) whether the bolt is useful or not. If we measure the length of each individual, the characteristic is quantitative and is said to be a variate. On the other hand, if we record the information in such a way that an individual (bolt) is classified as useful or not useful, then the characteristic is qualitative and hence is called an attribute.

Ex. 2. A municipality is interested in finding out whether the taximen in the city A, on an average, earn more than those in another city B, in a week.

A sample can be selected from the taximen in city A, their weekly earnings can be recorded and the average weekly earning can be found. Similar figures for the set of taximen in city B can also be found. Then these two averages can be compared to find the city in which the taximen earn more.

Here for city A, the population consists of all the taximen in that city. Each taximan in city A is an individual of the population. Total number of taximen in city A is the population size. The group of taximen selected from the population is the sample. Each one of these taximen selected is an individual of the sample, Total number of taximen so selected is the sample size. Characteristic to be studied is the weekly earnings of the taximen in the city. Since weekly earnings can be written down numerically, the characteristic is called a variate.

Ex. 3. For the following characteristics of individuals, let us decide whether they are variates or attributes :

Individual	Characteristic	Variate/ attribute
(i) A bulb	Life in hours	Variate
	Defective or not	Attribute
(ii) A student	Mother tongue	Attribute
	Sex	Attribute
	Marks in Statistics	Variate
	Age in years	Variate
	Religion	Attribute
	Weight in pounds	Variate
	Height in inches	Variate
(iii) A family	Number of earning members	Variate
	Income of the family	Variate
	Number of children	Variate
(iv) Employees of a company.	Monthly salary	Variate
	Smoker or not a smoker	Attribute
	Marital status	Attribute
	Wearing glasses or not	Attribute

§ 11·12. Types of variates :

As already observed, the number of children in a family is a variate and the height of each child is also a variate. But there is some difference in these two types of variates. The variate : number of children in the family, can take values such as 0, 1, 2, 3.....etc but not any value such as 2·5, 3·7 etc., i.e. in this case the variate takes only isolated values. Such a variate is called a discrete variate. In general, a variate which can take isolated values only is called a *discrete variate.*

On the other hand, the variate : the weight of a child, in a family can take any value such as 7·75, 8·90, 11·35 lbs....etc. Thus, it does not take only isolated values, but can take any value within a given range. Such a variate is called a continuous variate. In general, a variate which can take any value within a given range is called a *continuous variate.*

We will now find whether the variates considered above are discrete or continuous. The marks in statistics of students, the number of earning members and the number of children in a family, income of a family, monthly salaries of employees of a company are discrete variates, but the life in hours of a bulb, the weight in pounds and height in inches of students are continuous variates.

§ 11·2. Frequency Distributions :

§ 11·21. Frequency distribution of attributes :

Having already collected the information about the characteristic of an individual, we would like to know something more from the information collectively. If a set observations such as "P, F, F, P, F, P, F, P, P, P, ..." (where P = passed and F = failed) is given for 1000 students, it is difficult to get any idea regarding their performance. Such a data is known as raw data. But if we put the information in the form

P :	Passed	720
F :	Failed	280
	Total :	1000

then we can easily understand that 72% of the students have passed. Such a table is known as a frequency distribution and since the characteristic is an attribute in this case, this frequency distribution is called the frequency distribution of an attribute. The number 720 is called the frequency of the attribute P : 'passed' and 280 is the frequency of the attribute F : 'failed'.

Further $\frac{720}{1000} = 0{\cdot}72$ and $\frac{280}{1000} = 0{\cdot}28$ are called *relative frequencies* and 72% and 28% are known as *percentage frequencies* of the corresponding attributes. We can easily see that the sum of the frequencies is equal to the total number of individuals, the sum of the relative frequencies is one and the sum of the percentage frequecies is 100%. As you are already familiar with the method of preparing such frequency tables using tally marks, we give an example to revise the method.

Ex. The following data gives the results (P = passed, F = failed) of a group of 25 students :

P P F P P P P F P F F P P P
F P F P P F P F P F P

Prepare the frequency table and find the relative and the percentage frequencies and also their sums.

Ans.

Attribute	Tally Marks	Frequency	Relative frequency	Percentage frequency
Failed : F	𝍸 ||||	9	0·36	36%
Passed : P	𝍸 𝍸 𝍸 |	16	0·64	64%
Total		25	1·00	100%

Hence the sum of the relative frequencies is 1 and the sum of the percentage frequencies is 100%.

§ 11·22. Ferquency distribution of a discrete variate :

Now instead of simply giving, the results as F or P, if the marks of the students are given, as in the following example, we can prepare the frequency table showing frequencies corresponding to the various values of the variate.

Ex. The following data gives the marks (out of 5) of 25 students :

3 4 0 3 2 5 2 0 2 1 1 2 3 2 0
4 1 0 2 1 2 1 3 1 3

Prepare the frequency table.

Ans.

Marks	Tally Marks	Frequency
0	||||	4
1	𝍸 |	6
2	𝍸 ||	7
3	𝍸	5
4	||	2
5	|	1
		25

§ 11·23. Grouped frequency distribution :

If there are many values of a discrete variate, we can group them into convenient groups and prepare a table showing the groups and their frequencies. Such a table is known as a grouped frequency distribution. e. g. the marks (out of 100) in English obtained by the students of a class is a discrete variate which takes values from 0 upto 100. From the data of the marks of a group of 50 students of a certain school, suppose we are interested in finding how many of them have failed (marks less than 35), how many have passed (marks 35 or more) and out of those who have passed, how many have got first class (marks 60 or more), second class (45 and above but less than 60) etc.

In such a case, we can group the values of the variate into groups such as 0–34, 35–44, 45–59, 60–100 and prepare the frequency table with the help of these groups, as in the following example :

Ex. The following data gives the marks (out of 100) of a group of 50 students :

78 5 35 43 47 7 24 45 27 37
35 41 50 75 36 35 57 16 40 13
51 25 31 37 55 70 18 63 8 50
20 71 0 64 37 9 36 40 36 41
40 35 40 27 49 45 30 59 44 60

Prepare the frequency table, grouping the marks as 0–34, 35–44, 45–59, and 60–100.

Ans.

Marks	Tally Marks	Frequency
0–34	𝍸 𝍸 𝍸	15
35–44	𝍸 𝍸 𝍸 |||	18
45–59	𝍸 𝍸	10
60–100	𝍸 ||	7
		50

Thus, when the raw data containing the marks of the 50 students is put into a grouped frequency table as in the above example, it is very easy to get an idea about the number of students getting first class, second class etc. In this grouped frequency table, 0–34, 35–44,....etc. are called the *class intervals*. 45–59 is one of these class intervals in which 45 is the lower class limit and 59 is the upper class limit.

Just as in the frequency tables of attributes, we can find the relative and the percentage frequencies in this case also, e. g. relative frequency of the class interval 0–34 is $\frac{15}{50} = 0{\cdot}3$, whereas the percentage frequency for this class is $\frac{15}{50} \cdot 100 = 30\%$ which gives us the information that 30% of the students have failed or the proportion of students failing in the examination is $\frac{3}{10}$. This result is sometimes written as, ' on an average, 3 out of 10 students have failed in the examination. '

§ 11·24. **Requirements of class intervals :**

The class intervals should be such that each value of the variate must be included in *one* of the class intervals. For example, for preparing frequency table of the marks (out of 100), all the four intervals should be written because, if we write only three of these intervals, viz. 0–34, 45–59 and 60–100 then many values of the variates such as 35, 36, ... etc. are not included in any of these three intervals. Hence, the class intervals should be such that there must be a class-interval corresponding to each value of the variate.

Secondly, there should be *not more than one* class interval corresponding to any value of the variate, for example, class intervals such as 30–45 and 40–55 cannot be used because some values of the variate, viz. 40, 41, 42, ... etc. are now included in both these class intervals. Hence the class intervals should be disjoint, i. e. no value of the variate should be included in more than one class interval.

In short, ' the class intervals should be selected in such a way that corresponding to each value of the variate, there should be *one and only one* class interval '.

Ex. The following are the monthly salaries in rupees (rounded to the nearest rupee) of 25 employees of a firm :

220 245 257 239 271 273 262 235 279
292 205 289 263 243 213 299 241 283
255 267 260 310 267 275 269.

The firm gave bonuses of Rs. 20, 30, 40, 50, 60 and 70 etc. to the employees in the respective salary groups 200–219, 220–239,.. etc. Prepare (i) the frequency table for the salaries of the employees, (ii) frequency table for the bonus paid.

Ans. (i) Frequency Table of the Salaries :

Class Intervals	Tally Marks	Frequencies
200–219	||	2
220–239	|||	3
240–259	𝍸	5
260–279	𝍸 𝍸	10
280–299	||||	4
300–319	|	1
		25

(ii) Frequency table for Bonus paid :

Bonus	Frequency
20	2
30	3
40	5
50	10
60	4
70	1

§ 11·25. Frequency distribution of a continuous variate :

The class intervals of the form 30–34, 35–39,....etc., used in case of discrete variate cannot be used for preparing a frequency distribution of a continuous variate because a continuous variate can have some values in the interval 34-35 and there is no class interval where these values of the variate can be included. We cannot even use the class intervals such as 34–34·99, 35–39·99,.. etc. because even now there will be some values of the continuous variate between 34·99 and 35 which cannot be included in any of these class intervals. Further if we take the class intervals as 30–35 which includes both 30 and 35, and 35–40 which includes both 35 and 40,....then 35 is included in both class intervals. Obviously such class intervals cannot be used. Hence we use the class intervals of the form. '30 and more, but less than 35; 35 and more, but less than 40 ;... etc." Such class intervals will include all values of the variate such as 34·999, 34·99999 etc. and at the same time the value 35 will be included in one class interval only. It is customary to write these intervals as 30–35, 35–40, ..etc. which will mean the values:30 and above but less than 35, 35 and above but less than 40,....etc. respectively. Hence for every class interval the lower limit is included within the class and the upper limit is excluded from it. Hence these class intervals differ from the class intervals used for the discrete variate in that, the class interval 30–35 for a discrete variate includes both the class-limits, whereas the class-interval 30–35 used for a continuous variate includes 30 but does not include 35.

Using the class intervals in this form, we can prepare the frequency distribution of a continuous variate as in the following example and can also find the relative frequencies and the percentage frequencies as before.

Ex. The following data gives the amount of rainfall at a certain place during 1901–1950 :—

21·2, 29·1, 39·5, 41·7, 54·0, 50·8, 42·7, 45·1, 26·3, 36·48,
35·1, 32·3, 37·9, 52·0, 41·6, 39·5, 40·2, 33·3. 48·6, 33·51,
48·31, 34·24, 41·6, 25·1, 23·1, 36·3, 35·8. 42·0, 37·3, 28·5,
46·01, 31·31, 53·5, 34·8, 39·3, 54·2, 37·1, 46·0, 33·0. 44·43,
30·31, 44·5, 47·45. 27·31, 38·4, 45·4, 36·8, 43·7, 52·8, 42·55.

Prepare the frequency table taking class-intervals 20–25, 26–30, ... etc. Find also the relative and percentage frequencies.

Ans.

Class-intervals	Tally Marks	Frequency	Relative frequency	% frequency
20–25	||	2	0·04	4
25–30	𝍸	5	0·1	10
30–35	𝍸 |||	8	0·16	16
35–40	𝍸 𝍸 ||	12	0·24	24
40–45	𝍸 𝍸	10	0·2	20
45–50	𝍸 ||	7	0·14	14
50–55	𝍸 |	6	0·12	12
Total		50	1·00	100

§ 11·26:

When the class-intervals of the form mentioned here are used for a continuous variate, the *length* or *width* of the class-interval is the difference between the upper and the lower class-limits and *mid point* of a class-interval is the average of the upper and lower class-limits. Mid point of a class-interval is also known as the *class-mark* of the class-interval, e. g. for the class-intervals 30–35, 35–40, 40–45, ...etc. mentioned above, length of the class interval 30–35 is $35-30=5$; length of the interval 35–40 is also 5 ($=40-35$). Class-mark or mid point of the class-interval $30-35$ is $\frac{30+35}{2}=\frac{65}{2}=32{\cdot}5$ and of $35-40$ is $\frac{35+40}{2}=\frac{75}{2}$ $=37{\cdot}5$.

Sometimes, when any two of (i) mid points or (ii) the length of the class-interval, or (iii) lower and upper class-limits are known, we can write down the class-intervals, e. g. if the

length of the class-interval is 10 and the mid point is 25, the class-interval becomes 20–30. Similarly, if the width of a class-interval is 5 and the class-mark is 40, the class-interval is 37·5 — 42·5. This is used in the following example.

Ex. The lengths in inches of 50 bars are as follows :—

106 82 129 204 90 107 123 139 111 107 76 125 119
141 81 82 111 115 136 131 109 92 128 123 75 107
86 101 90 84 93 126 84 98 110 187 68 99 110
81 95 130 113 78 118 115 70 186 115 104

Form a grouped frequency table by dividing the variate range into intervals of equal width, each corresponding to 20″, in such a way that the mid-value of the first class corresponds to 70″.

Ans. Since the width of the class-interval is 20″ and the class mark of the first interval is 70″, the lower class limit is $70 - 10 = 60$ and the upper limit is $70 + 10 = 80$. Hence the class intervals are 60–80, 80–100....etc. The frequency table is as follows :

Class intervals (lengths in inches of bars)	Tally Marks	Frequency
60–80	𝍸	5
80–100	𝍸 𝍸 ||||	14
100–120	𝍸 𝍸 𝍸 ||	17
120–140	𝍸 𝍸	10
140–160	|	1
160–180		0
180–200	||	2
200–220	|	1
Total		50

§ 11·27. Continuous variate recorded as discrete :

We have already observed that the class intervals of the form 30–34 and 35–30,....etc. cannot be used for a continuous variate because a continuous variate may take values such as 34·1, 34·5, 34·9,....etc. However, in practice, sometimes a continuous variate may be recorded as a discrete variate by rounding off the values, preferably to the nearest integer, for example, 34·1 may be recorded as 34, 34·9 as 35 and 34·5 also as 35.

Having recorded the values of a continuous variate as if it is a discrete variate, we can naturally use the class intervals of the form 30–34, 35–39,....etc. for preparing a frequency table of this variate. Here the class intervals 30–34 will include the values of the continuous variate 34·1, 34·2, 34·49,....and also 29·5, 29·7, 29·99,... etc. as these two sets of values would be recorded as 34 and 30 respectively. Similarly, all the values of the variate between 34·5 and 35 including 34·5 will be recorded as 35 and will now be included in the class interval 35–39. Similarly all the values of the variate between 39 and 39·5 excluding 39·5 will be recorded as 39 and will also be included in the same class interval, viz. 35–39. Hence the class interval 35–39 will, in fact, include all the values of the variate between 34·5 and 39·5 including 34·5 and excluding 39·5, i. e. the class interval 35–39, in fact, represents the class interval 34·5–39·5, 34·5 and 39·5 are called the *class boundaries* of the class interval 35–39. 34·5 is called the lower class boundary and 39·5 is the upper class boundary. Similarly the class boundaries for the class interval 30–34 are 29·5 and 34·5 respectively.

In such a case the length or the width of the class interval of the continuous variate is the difference between the upper and the lower class boundaries, e. g. for the class intervals 30–34, 35–39,....etc. considered in the above example, the length of the class interval of 30–34 is $34{\cdot}5 - 29{\cdot}5 = 5$; of the class interval 35–39 is $39{\cdot}5 - 34{\cdot}5 = 5$,....etc.

The mid-point of the class interval is the average of the upper and the lower class boundaries, e, g. the mid-point of the class interval 30–34 is $\frac{29{\cdot}5 + 34{\cdot}5}{2} = \frac{64}{2} = 32$.

In case of the frequecy table consisting of eqaul-length class-intervals, the average of the class-boundaries will be the same as the average of the class-limits, hence in such cases we can find the mid point by taking the average of the class-limits, e. g. the mid-points of the class interval 30–34 is $\frac{30+34}{2}=\frac{64}{2}=32$.

Note that when the class intervals of the form 30–34, 34–38, 38–42,....etc. are used for preparing a frequency table for a continuous variate the class intervals and the class boundaries coincide and hence in such a case the length of the class interval can be the difference between the upper and the lower class limits.

§ 11·28. Cumulative frequency :

Consider the following distribution of marks (out of 50) obtained by students ;—

Marks	0–10	10–20	20–30	30–40	40–50
Number of students	8	15	29	22	6

Suppose we ask : " How many students obtained less than 30 marks?" This can be answered by adding the frequencies of the classes 0–10, 10–20 and 20–30, i. e. the answer is $8+15+29=52$.

Hence the total frequency of all values from 0 to 30, but not including 30, is 52. This is called the cumulative frequency of 30. Since this cumulative frequency is the frequency of all values less than 30, it is called cumulative frequency (less than type or $<$ type).

Similarly, the frequency of values greater than or equal to 30 is $22+6=28$. This is called the cumulative frequency ($\geq$ type) of 30.

Hence, in general, we can say that :

(1) If x_r is a value of the variate x, the frequency of all values of the variate which are less than or equal to x_r is called the cumulative frequency ($<$ type) of x_r.

(2) If x_r is a value of the variate x, the frequency of all

values of x which are greater than or equal to x_r is called the cumulative frequency ($\geqslant$ type) of x_r.

In the case of grouped frequency distribution, if the class intervals are $a_0 - a_1$, $a_1 - a_2$, $a_2 - a_3$, ... with frequencies f_1, f_2, f_3, ... respectively then the cumulative frequencies are to be determined corresponding to the end points only, i. e. cumulative frequency ($<$ type) of a_3 will be $f_1 + f_2 + f_3$, whereas the cumulative frequency ($\geqslant$ type) for a_2 is $f_3 + f_4 + \ldots + f_n$ where f_n is the frequency of the last class-interval, e. g. consider the following distribution :—

Weight of students (lbs.)	70–80	80–90	90–100	100–110	110–120	120–130
Number of students	12	17	37	25	18	6

Then the table showing both types of cumulative frequency is :

x_i	Cumulative frequency ($<$ type) $f < x_i$
70	0
80	12
90	29
100	66
110	91
120	109
130	115

x_i	Cumulative frequency ($\geqslant$ type) $f \geqslant x_i$
70	115
80	103
90	86
100	49
110	24
120	6
130	0

§ 11·3. Frequency diagrams :

Classification of data into a frequency distribution is the first step in summarising the available information. A frequency diagram helps us to present this information graphically. Such a

graphical representation of a frequency distribution gives a better picture of the frequency distribution.

The two standard methods of graphically representing a frequency distribution are (1) frequency polygon and (2) Histogram.

§ 11·31. Frequency Polygon :

For a drawing a frequency polygon for the frequency distribution of a variate x, we take the values of the variate along the horizontal axis, known as x axis, and the frequency along the vertical axis. The vertical scale must always start with zero, whereas the horizontal scale should include the range of the values of x and one extra class-interval at both ends.

Let $x_1, x_2, \ldots . x_n$ be the class-marks of the class-intervals whose frequencies are $f_1, f_2, \ldots . f_n$. Plot the points (x_1, f_1), $(x_2, f_2), \ldots . (x_n, f_n)$ on the graph.

Take one more class-interval at each end. Let x_0 and x_{n+1} be these class-marks. Since the frequencies of these two class intervals are zero, the corresponding points are $(x_0, 0)$ and $(x_{n+1}, 0)$ Plot these also on the graph.

Joining the consecutive points by straight lines, we get the diagram of the frequency distribution known as a frequency polygon. We shall now give an example to illustrate the method.

Class-Intervals	Frequency f	Class-mark x
10–20	5	15
20–30	8	25
30–40	13	35
40–50	10	45
50–60	4	55
	40	

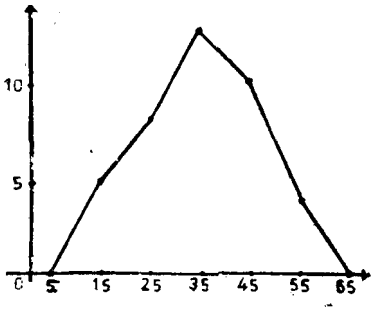


Fig. 11·1

The points (x_i, f_i), viz. (15, 5), (25, 8), (35, 13), (45, 10), and (55, 4) are plotted on the graph. Further we take one more interval at each end, viz. 0–10 and 60–70. Their class-marks are 5 and 65 and the frequency of each is zero. Hence we plot the points (5, 0) and (65, 0). Joining the consecutive points by straight lines, as shown in fig. 11·1, we get the frequency polygon of the given frequency distribution.

§ 11·32. Histogram :

When the values of a continuous variate are put in the form of a grouped frequency distribution, we can draw rectangles whose widths are proportional to the lengths of the class intervals and whose areas are proportional to the frequencies of the corresponding class-intervals. Such a diagram is called the *Histogram* for the given frequency distribution.

Equal-length class-intervals :

When the class-intervals are of equal length, the width of the rectangles will be the same and hence the heights of the rectangles can be taken proportional to the frequency directly. The Histogram for the following frequency distribution, whose class-intervals are of equal length, is shown in fig. 11·2.

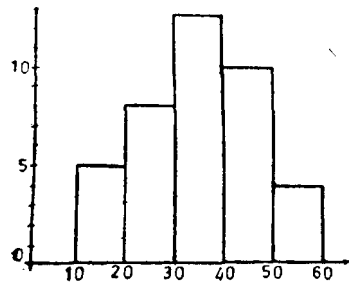


Fig. 11·2

Class-intervals	Frequency
10–20	5
20–30	8
30–40	13
40–50	10
50–60	4
	40

Since the length of each class-interval is the same, the width of these rectangles is the also the same and further, since the heights of the rectangles are taken to be proportional to the frequency, the width being equal, the areas (height × width) are proportional to the frequencies of the class-intervals.

Unequal class intervals:

In case of a frequency distribution of a continuous variate where the class intervals are not of equal length, the width of the rectangles can be taken proportional to the length of the class-interval whereas the heights of the rectangles can be taken proportional to $\frac{\text{frequency of a class interval}}{\text{length of the class interval}}$.

Since the area of the rectangles, i. e. product of height and width of the rectangles is proportional to the frequency, i. e. height of rectangle $\propto \frac{\text{frequency}}{\text{width of the class-intervals}}$.

This is illustrated in the following example :

Class-intervals	Frequency f	Length or width of class-intervals l	Height $h=\frac{f}{l}$
10–20	10	10	1
20–40	60	20	3
40–50	60	10	6
50–80	90	30	3
80–120	80	40	2
	300		

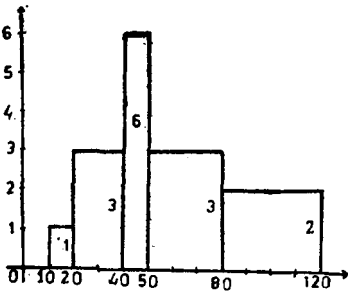


Fig. 11·3

With the heights taken along the vertical and the class intervals along x-axis, we can draw the Histogram for this data, as shows in fig. 11·3.

Continuous variate recorded as discrete :

In case of a grouped frequency distribution where a continous variate is recorded as a discrete variate and the class intervals of the form 10–19, 20–29....etc. are used then, for drawing a Histogram, the class boundaries should be taken, i. e. the class limits 10–19, 20–29, ... etc. should be taken as 9·5 – 19·5. 19·5 – 29·5, ... etc. and then with the help of these class boundaries the Histogram should be drawn.

§ 11·33. Frequency Polygon and Histogram :

Histogram and frequency polygon are helpful in presenting frequency distribution graphically, with the help of which we can find out how the values of the variate are distributed. In fact, both these tell the same story. However, each of them has its own advantages.

Histogram is more exact than the frequency polygon in the sense that the area included in each retangle represents the exact number of values in the corresponding class–interval. The individual classes can be observed more clearly in a Histogram than in a frequency polygon.

On the other hand, the frequency polygons of two frequency distributions can be easily compared than their Histograms. Further, frequency polygons give a clearer conception about the shape of the curve.

In general, when the class intervals are few, a Histogram is preferable whereas, when the class–intervals are many, a frequency polygon is preferable.

Comparison of two frequency distributions : By taking the relative frequencies or percentage relative frequencies of two frequency distributians and drawing their frequency polygons on the same scale, it is easy to compare two different sets of data or two different frequency distributions.

§ 11·34. Frequency Curve :

A frequency curve can be considered as a limiting form of either the frequency polygon or the Histogram as the size of the sample becomes indefinitly large and the class intervals becomes infinitely small. As the idea of frequency curves includes the concepts of sampling theory, we shall not go into much detail. Here we only consider it as a smooth curve passing through the points (x_i, f_i); $i = 1, 2, 3, \ldots$ used in drawing the frequency polygon. This method is known as *smoothing of frequency polygon* and is illustrated in fig. 11·4. The curve obtained is called a **frequency curve.**

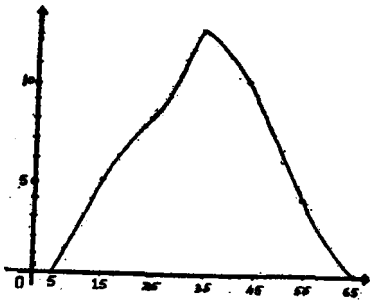


Fig. 11·4

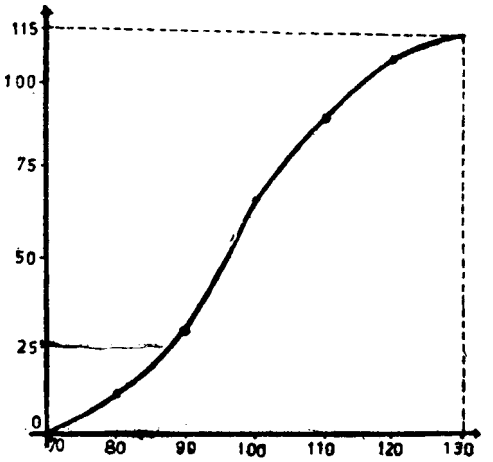
§ 11·35. Cumulative frequency curves or ogives :

Just as a frequency polygon can be drawn by taking x_i on the x axis and f_i on the vertical axis, similarly, if we take x_i on the x axis and the corresponding cumulative frequencies ('less than type' or 'greater than or equal to type') on the vertical axis, and draw a smooth curve passing through these points, we get a diagram known as a cumulative frequency curve or an ogive.

We have already found cumulative frequencies of less than type and cumulative frequencies of greater than or equal to type for a given frequency distribution in § 11·28. We however, repeat the tables containing these values and then draw the two curves known as ogive (less than type) and ogive (greater than or equal to type) as follows :

Weight of students (lbs.)	70–80	80–90	90–100	100–110	110–120	120–130
Number of students	12	17	37	25	18	6

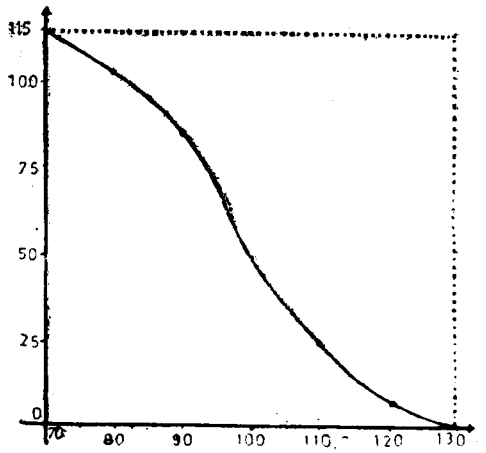
(1) Cumulative frequency curve or ogive (less than type):



x_i	$f < x_i$
70	0
80	12
90	29
100	66
110	91
120	109
130	115

Fig. 11·5

(2) Cumulative frequeney curve or ogive (greater than or equal to type).



x_i	$f \geq x_i$
70	115
80	103
90	86
100	49
110	24
120	6
130	0

Fig. 11·6

Exercise 11·1 :

(1) Referring to an example, explain the following terms: (a) population, (b) individual of the population, (c) variate.

(2) Distinguish between (a) variate and attribute; (b) continuous and discrete variate and give suitable examples in each case.

(3) Explain what is meant by: (a) frequency distribution, (b) grouped frequency distribution, (c) class mark, (d) class boundaries, (e) width of a class-interval, (f) cumulative frequency.

(4) For each of the following, state whether it is a variate or an attribute. In case it is a variate, state whether it is discrete or continuous.

(a) weights of students, (b) mother tongue of students, (c) speeds of a car, (d) rainfall in inches at a place during a century, (e) number of cars passing a junction during the daylight hours, (f) number of thefts per week in a city during a year.

(5) The following data gives the marks in statistics obtained by forty students:

47 32 26 43 33 21 34 28 31 33 25 43 47 35
23 27 31 41 46 33 36 32 31 27 21 38 41 22
31 39 33 37 42 23 22 28 27 31 34 30.

Prepare a frequency table by taking class-intervals 20–24, 25–29, ... etc.

(6) The following data gives the number of people living in each of fifty buildings:

21 50 35 39 48 46 36 54 42 30 29 42 32 40 34
31 35 37 52 44 39 45 37 33 41 53 42 46 43 47
41 26 52 48 25 34 37 33 36 27 54 36 41 33 23
39 28 44 45 38.

Prepare a frequency-table taking class-intervals 20-24, 25–29, 30–34, ... etc. Draw a histogram for the distribution.

(7) The following table gives the June rainfall, in inches, in Bombay for 78 years (1857 to 1934):

9 14 27 22 15 22 23 15 11 13 9 14 26 21 15
18 14 13 5 43 24 16 20 22 9 24 20 19 24 13
35 20 17 25 14 13 21 17 18 28 14 27 21 18 25
10 20 15 7 13 22 15 27 24 11 11 26 17 40 24
15 11 23 8 27 27 9 7 27 7 36 28 26 27 8
11 10 29.

Prepare a frequency table with class-intervals 5–10, 10-15,.... etc. Draw the histogram and the frequency polygon.

(8) The following data gives the weekly earnings of 50 taximen in Bombay, selected at random, in rupees (rounded to the nearest integer) :

72	85	93	84	105	121	71	89	92	106
107	112	78	96	104	103	87	76	85	104
92	103	92	101	89	99	104	86	103	102
74	86	97	94	91	76	82	128	124	104
103	102	101	131	102	117	104	112	139	71.

Prepare a frequency-table taking class intervals 71–75, 76–80,.... etc. Draw the frequency polygon.

(9) The following figures give the July-rainfall in Bombay for the years 1894 to 1962 :

22	26	18	22	26	7	25	10	29	23	32	33	16	15
25	36	27	24	17	13	26	31	31	17	40	25	34	42
26	10	26	31	17	18	41	22	18	15	14	8	40	15
36	36	18	26	31	29	24	31	30	59	13	14	20	11
22	21	21	4	24	30	30	5	31	24	11.			

Prepare a grouped frequency table with class-intervels 0–7, 8–15, 16–23 etc.

Draw the histogram and the frequency polygon of the distribution.

(10) The lengths (in inches) of 50 bars are as follows :

106	82	129	204	90	107	123	139	111	107
76	125	119	141	81	82	111	115	136	131
109	92	128	123	75	107	86	101	90	84
93	126	84	98	110	187	68	99	110	81
95	130	113	78	118	115	70	186	115	104.

Form the grouped frequency table by dividing the variate range into intervals of equal width, each corresponding to 20", in such a way that the mid-value of the first class corresponds to 70". Prepare the cumulative frequency table of less than type and draw the ogive.

(11) The following data gives the lengths, recorded to the nearest inch, of 30 iron bars :

95	106	107	76	109	107	115	93	117	100
66	92	95	97	67	92	93	72	75	68
110	122	123	117	121	104	134	126	102	103.

Prepare a frequency distribution-table with class-intervals 66–80, 81–95, ... State the class boundaries and the class mark of the interval 81–95. How many bars have length greater than 95 inches ?

(12) Obtain the cumulative frequency table of both types from the following data showing percentages of marks scored in the annual examination :

Percentage of marks	Number of students
0–10	3
10–20	11
20–30	34
30–40	47
40–50	55
50–60	39
60–70	23
70–80	10
80–90	3
90–100	1

Draw the less than type ogive for this distribution.

(13) The following table gives the earnings (in rupees per month) of 70 men employed in a mill.

72	86	94	84	105	120	71	89	92	106
107	112	78	96	104	103	87	76	85	104
92	103	92	101	89	99	104	86	103	102
75	86	98	95	92	77	82	127	124	103
103	102	100	130	102	117	104	112	100	72
112	104	109	110	112	93	87	74	79	81
84	102	101	97	107	101	104	105	103	94.

Prepare a frequency table taking class intervals 71–75, 76–80, 81–85, ... State the class boundaries of the interval 81–85.
Prepare the cumulative frequency table of less than type and draw the ogive.

(14) The following is a dirtribution of marks obtained by 100 students:.

Marks	Number of students
0–10	5
10–20	9
20–30	13
30–40	28
40–50	20
50–60	12
60–70	8
70–80	3
80–90	1
90–100	1

(a) Prepare the cumulative frequency ($<$ type) table and draw the ogive.

(b) Answer the following questions:

(i) How many students have secured 30 or more marks?

(ii) What percentage of students gets 30 or more but less than 60 marks?

(iii) What percentage of students get less than 20 marks?

(iv) What is the mark above which we get the brightest 25% students?

(v) What is the mark below which we have the worst 27%?

(vi) Which range of 30 marks has the largest number of students?

(vii) If 30 is the mark for passing, how many students have failed?

(viii) If first class is awarded to a student scoring 60 or more marks, what is the percentage of students getting first class?

CHAPTER 12

Measures of Location and Dispersion

§ 12·1. Measures of Location :

While studying a statistical data, the grouping of the data into frequency distribution and the graphical representation of the same by a histogram or a frequency polygon does not yield sufficient information. This is especially true if we want to compare two sets of data. In such a case, merely finding the frequency table is not sufficient. Hence we must find some quantitative measures based on a frequency distribution which will give more precise information about the characteristics of the distribution.

If the observations are such that a large number of them are clustered around a small interval about a certain value, this value is very important, as it can be treated as an approximation of a large number of observations. Finding such a value is called finding a measure of location.

Some standard measures of location are given below:

(1) **Arithmetic mean** : If $x_1, x_2, x_3, \ldots, x_n$ are n given observations, their arithmetic mean $\bar{x}$ is defined as

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$ $\bar{x}$ is generally called the **mean.**

(2) **Geometric mean** : If $x_1, x_2, x_3, \ldots, x_n$ are n observations which are all positive, their geometric mean is defined as

$$\sqrt[n]{x_1 \cdot x_2 \cdot x_3 \ \ldots \ x_n} = (x_1 \ x_2 \ x_3 \ldots . x_n)^{\frac{1}{n}}$$

e. g. Geometric mean of 3 and 27 is $\sqrt{3 \times 27} = 9$ and geometric mean of 4, 8 and 16 is $\sqrt[3]{4 \times 8 \times 16} = 8$.

(3) **Harmonic mean** : If $x_1, x_2, x_3, \ldots, x_n$ are n observations, none of which is zero, then their harmonic mean H is defined by

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)$$

$$\therefore \quad H = \frac{n}{\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

e. g.

Harmonic mean of 30 and 60 is $\frac{2}{\frac{1}{30} + \frac{1}{60}} = \frac{2}{\left(\frac{3}{60}\right)} = 2\,(20) = 40.$

(4) **Median**: If the observations $x_1, x_2, \ldots . x_n$ are arranged either in increasing or in decreasing order of magnitude and if (i) n is odd then the middle observation is the median and if (ii) n is even, then the arithmetic mean of the two middle observations is the median.

(5) **Mode**: The observation that occurs most frequently (i. e. the maximum number of times) in the given data is called the mode.

Note: The measures of location (1) to (4) always exist but the mode may not exist or may not be unique.

§ 12·11. Examples:

Ex. 1. Find the mean and median of the observations:

1, 2, 3, 2, 3, 4, 1, 3, 1, 2, 3, 5.

Ans. The mean

$$= \frac{1+2+3+2+3+4+1+3+1+2+3+5}{12} = \frac{30}{12} = 2.5$$

To find median, we arrange the observations in ascending order as follows:

1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 5.

Since there are twelve values, the median is the mean of the sixth and seventh terms, i. e. 2 and 3.

$$\therefore \quad \text{Median} = \frac{2+3}{2} = 2{\cdot}5$$

Ex. 2. Find the mean, median and mode of the values:

4, 1, 3, 2, 3, 4, 3, 3, 1, 2, 5, 2, 0, 1, 6.

Ans. Mean

$$= \frac{4+1+3+2+3+4+3+3+1+2+5+2+0+1+6}{15}$$

$$= \frac{40}{15} = 2\frac{2}{3}.$$

The values, arranged in order, are :

0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 6.

There are 15 in number. ∴ The median is the eighth term which is 3. ∴ median = 3.

Mode = 3, since 3 occurs maximum (i. e. 4) times.

Exercise 12·1 :

(1) Find the mean, median and mode of the following data :

(a) 1, 1, 2, 1, 3, 1, 1, 4, 5, 3, 3, 2, 1, 2, 2.
(b) 5, 6, 6, 5, 4, 5, 3, 2, 5, 2.
(c) 1, 2, 1, 2, 1, 3, 1, 5, 4, 7, 2, 5, 3, 2, 3, 2.
(d) 11, 11, 12, 11, 13, 12, 11, 14, 12, 13.
(e) 618, 619, 616, 623, 620, 622, 625, 622, 625, 626.

(2) Find k if the mean of the following set of values is 4 :
2, 3, 7, 3, 5, 7, 2, k, 1, 5.

§ 12·12. Mean for a frequency distribution :

In case of frequency distribution of a variate, we can find the mean as follows :

Suppose a value x_1 has frequency f_1. Then x_1 is repeated f_1 times. Hence when we add all the values, x_1 has to be added f_1 times. But $x_1 + x_1 + x_1 \ldots . f_1$ times $= x_1 \cdot f_1 = f_1 x_1$.

Similarly if x_2 has frequency f_2, then the addition of these f_2 values, each equal to x_2, is $f_2 x_2$.

Hence if the variate x takes values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively, then the sum of all these values is $f_1 \cdot x_1 + f_2 \cdot x_2 + f_3 \cdot x_3 + \ldots + f_n \cdot x_n$.

Also the total number of values is $f_1 + f_2 + f_3 + \ldots + f_n$.

Hence the mean $\bar{x}$ will be defined as follows :

Definitions : If a variate x takes values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively, then the mean $\bar{x}$ is given by

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \ldots + f_n x_n}{f_1 + f_2 + \ldots + f_n}$$

Using the sigma notation explained in Algebra, this can be written, in short, as

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} .$$

In solving an example, since the frequency distribution has columns of x_i and f_i, we prepare a third column of $f_i \cdot x_i$ and find the totals of the values in the required columns, as in the following example :

Ex. Find the mean of the number of children in a family from the follcwiag data :

No. of children	0	1	2	3	4	5
No. of families	5	8	23	10	3	1

Ans.

No. of children x_i	No. of families f_i	$f_i x_i$
0	5	0
1	8	8
2	23	46
3	10	30
4	3	12
5	1	5
Total	50	101

Here $\Sigma f_i x_i = 101$ and $\Sigma f_i = 50$

$$\therefore \quad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{101}{50} = 2{\cdot}02.$$

If a grouped frequency distribution is given, we treat the class mark of each class interval as x_i for that class interval and proceed as in the following example.

Ex. Find the mean of the marks obtained by students from the following data :

Marks	0–20	20–40	40–60	60–80	80–100
No. of students	8	12	15	10	5

Ans.

Marks	Class Mark x_i	No. of students f_i	$f_i x_i$
0–20	10	8	80
20–40	30	12	360
40–60	50	15	750
60–80	70	10	700
80–100	90	5	450
Total		50	2340

Here $\Sigma f_i x_i = 2340$ and $\Sigma f_i = 50$

$$\therefore \quad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{2340}{50} = 46{\cdot}8.$$

§ 12·13. Change of origin and scale :

Consider the following distribution giving the weights of students in a class :

Weight of student in lbs. (x_i)	141	143	145	147	149	151	153	155	157	159
Number of students (f_i)	1	3	8	25	32	21	17	7	6	0

If we want to find the mean weight of the students by the method explained in § 12·12, it will involve very difficult calcula-

tions as we have to find products like 149×32, 151×21, ...etc. and then add such numbers. Hence, the calculation becomes tedious, particularly when the values of the variate involves numbers involving decimals such as 149·99, 243·1257, etc.

Now suppose we define a new variable u taking values $u_1, u_2, u_3 \ldots$. where $u_i = x_i - 149$.

These u_i will have the values $-8, -6, -4, -2, 0, 2, 4, 6, 8, 10$ which are comparatively very small. Using these values of u_i in place of x_i, we can ensily calculate the mean $\bar{u}$ of u. We must find out the formula to calculate $\bar{x}$ from $\bar{u}$ and then the calculation of $\bar{x}$ is a very simple matter. This is called *change of origin* to 149.

Also note that all values of u_i are multiples of 2. Hence if we were to define $u_i = \frac{x_i - 149}{2}$, the values of u_i will be $-4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ which are still simpler. This is called *changing the scale* to half that of the original one.

Hence in short, if x is a variate taking large values $x_1, x_2, \ldots, x_n$ and if we choose numbers a and c ($c \neq 0$) such that $u_i = \frac{x_i - a}{c}$, $i = 1, 2, \ldots, n$ then if u_i are small numbers. it is easy to calculate $\bar{u}$. We now give the formula to calculate $\bar{x}$ with the help of $\bar{u}$.

Theorem 12·1. If a variate x takes values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively and if we define $u_i = \frac{x_i - a}{c}$ $a, c \in R$ and $c \neq 0$ for $i = 1, 2, 3, \ldots, n$ then $\bar{x} = a + c\bar{u}$, where $\bar{x}$ and $\bar{u}$ denote the means of the variates x and u respectively.

Proof: We know that $\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$ and $\bar{u} = \frac{\sum_{i=1}^{n} f_i u_i}{\sum_{i=1}^{n} f_i}$

Now $u_i = \frac{x_i - a}{c}$, $\therefore \quad x_i - a = cu_i$

$\therefore \quad x_i = a + cu_i$

$\therefore \quad f_i x_i = af_i + c f_i u_i$

$$\therefore \quad \sum_{i=1}^{n} f_i x_i = \sum_{i=1}^{n} a f_i + \sum_{i=1}^{n} c f_i u_i$$

$$= a \sum_{i=1}^{n} f_i + c \sum_{i=1}^{n} f_i u_i$$

$$\therefore \quad \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = a \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} f_i} + c \frac{\sum_{i=1}^{n} f_i u_i}{\sum_{i=1}^{n} f_i}$$

$$\therefore \quad \bar{x} = a + c\,\bar{u}$$

§ 12·14. Examples:

Ex. 1. Find the mean weight of students from the data given in § 12·13.

Ans.

Weight of student	Number of students	$u_i = \frac{x_i - 149}{2}$	$f_i u_i$
141	1	−4	−4
143	3	−3	−9
145	8	−2	−16
147	25	−1	−25
149	32	0	0
151	21	1	21
153	17	2	34
155	7	3	21
157	6	4	24
159	0	5	0
Total	120	—	−54 + 100 = 46

Hence $\Sigma f_i = 120$ and $\Sigma f_i u_i = 46$

$$\therefore \quad \bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i} = \frac{46}{120} = \frac{23}{60}$$

Now since $x_i = 149 + 2u_i$

$$\therefore \quad \bar{x} = 149 + 2\ \bar{u}$$

$$\therefore \quad \bar{x} = 149 + 2\left(\frac{23}{60}\right) = 149 + \frac{23}{30}$$

$$= 149 + 0{\cdot}766 \text{ approximately}$$

$$\therefore \quad \bar{x} = 149{\cdot}766.$$

Ex. 2. Find the mean for the following distribution :

Class Intervals	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40
Frequency	2	5	7	13	21	16	8	3

Ans.

Class-Intervals	Class marks x_i	Frequency f_i	$u_i = \frac{x_i - 22{\cdot}5}{5}$	$f_i u_i$
0–5	2·5	2	−4	−8
5–10	7·5	5	−3	−15
10–15	12·5	7	−2	−14
15–20	17·5	13	−1	−13
20–25	22·5	21	0	0
25–30	27·5	16	1	16
30–35	32·5	8	2	16
35–40	37·5	3	3	9
Total		75	—	−50+41 = −9

Hence $\Sigma f_i = 75$ and $\Sigma f_i u_i = -9$

$$\therefore \quad \bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i} = \frac{-9}{75} = \frac{-3}{25} = -0{\cdot}12$$

Now $x_i = 22{\cdot}5 + 5u_i$

$\therefore \quad \bar{x} = 22{\cdot}5 + 5\bar{u} = 22{\cdot}5 + 5(-0{\cdot}12) = 22{\cdot}5 - 0{\cdot}6$

$\therefore \quad \bar{x} = 21{\cdot}9$

Ex. 3. The mean monthly salary of the employees in a company is Rs. 96/-. The mean monthly salary of the males is Rs. 100/- and that of the females is Rs. 80. Find the percentage of males and females in the company.

Ans. Let there be m males and f females employed in the company.

Hence the total number of employees $= m + f$.

Since the mean salary of the employees is Rs. 96, the total salary paid per month $= 96(m+f)$ .... (i)

Now, since the mean salary of a male is Rs. 100, the total salary paid to males $= 100\,m$.

Similarly, the total salary paid to females $= 80\,f$.

$\therefore$ Total salary paid per month $= 100\,m + 80\,f$ .... (ii)

From (i) and (ii), $100\,m + 80\,f = 96(m+f)$

$\therefore \quad 100\,m + 80\,f = 96\,m + 96\,f \quad \therefore \quad 4\,m = 16\,f \quad \therefore \quad m = 4f.$

$\therefore$ The number of males is 4 times that of females, i. e. the proportion of males to females is 4 : 1. Hence 80 % employees are males and 20 % are females.

Exercise 12·2 :

(1) Find the mean for the following distributions :

(a)

x_i	−2	−1	0	1	2
f_i	5	8	13	10	4

(b)

x_i	1	2	3	4	5
f_i	7	12	22	10	9

(c)

x_i	2	4	6	8	10
f_i	5	8	10	5	2

(2) The following distribution gives the height of 350 men. Calculate the mean height :

Height (inches)	59	61	63	65	67	69	71	73
Number of men	1	2	9	48	131	102	40	17

(3) The annual salaries of a group of employees are given below. Find the mean salary.

Salaries (in hundreds of rupees)	45	50	55	60	65	70	75	80
Number of persons receiving the amount	3	5	8	7	9	7	4	7

(4) The daily salaries of workers are given below. Find the mean salary.

Salary (in rupees)	5·0	5·5	6·0	6·5	7·0	7·5	8·0
Number of workers	26	30	80	128	70	51	15

(5) Find the mean of the following distribution :

Class interval	3–5	5–7	7–9	9–11	11–13	13–15	15–17
Frequency	5	8	30	82	45	24	6

(6) The wages per month of 500 employees of a firm are shown below. Find the mean wages.

Wages (Rs.)	0–50	50–100	100–150	150–200	200–250	250–300
Number of employees	90	150	100	80	70	10

(7) The following table gives the chest girth in cms. of 30 men. Find its mean.

Chest girth	50-54	55-59	60-64	65-69	70-74	75-79	80-84	85-89	90-94
Number of men	3	2	3	2	1	4	9	5	1

(8) If the mean of the following distribution is 5·92, find the missing frequency f of observation 7.

x_i	4	5	6	7	8
f_i	8	10	15	f	5

(9) The mean wage of n workers is $\bar{x}$. If the sum of the wages of the first $(n-1)$ workers is s, show that the wage of the nth worker is $(n\bar{x}-s)$.

(10) The mean annual salary paid to all employees in a factory is Rs. 50,000/-. If the mean annual salaries paid to the male and female employees are Rs. 52,000/- and Rs. 42,000/- respectively, find the percentage of males and females employed.

(11) The mean age of a class is 13·25 years. The mean age of the boys in the class is 14 years and that of the girls is 12 years. If the number of boys exceeds the numher of girls by 10, find the number of boys and girls in the class.

§ 12·15. Median and quartiles for a frequency distribution :

We know that the median of a set of n observations arranged either in ascending or in descending order is the middle value if n is odd and the mean of the two middle values if n is even.

Hence, median divides the total frequency in two equal parts, 50 % of the total frequency preceding it and 50 % succeeding it.

Now we define quartiles called Q_1 and Q_3 as follows: The quartile Q_1 is the observation such that 25 % of the observations are below it and 75 % are above it. The quartile Q_3 is the observation such that 75 % of the observations are below it and 25% are above it. Let us see how these can be calculated from a frequency distribution :—

(I) Frequency distribution (ungrouped) :

Consider the data :

x_i	f_i	Cumulative frequency (less than type)
1	5	5
2	8	13
3	23	36
4	10	46
5	4	50
	50	

Since the total number of observations is 50, the median is the mean of the 25th and 26th observation.

Obviously both these observations are equal to 3 since the cumulative frequency upto 2 is 13 which means there are 13 observations less than and including 2 and the observations from the 14th to the 36th are all equal to 3 Hence median $= \frac{3+3}{2} = 3$.

(II) Grouped frequency distribution (continuous variate) :

Consider now the following frequency distribution :

Class interval	Frequency	Cumulative frequency < type
100–110	5	5
110–120	8	13
120–130	20	33
130–140	10	43
140–150	7	50
	50	

In case of continuous variate, the median is the value of the variate such that half of observations $\left(\text{i. e. } \frac{n}{2}\right)$ are below it and other half are above it. Hence, we shall find the point where the $\frac{n}{2}$ th observation ends. Let $m = \frac{n}{2}$. $\therefore m = \frac{50}{2} = 25$. Since these are class intervals and not discrete values, we can find only the class-interval which contains this point. Since 13 values are below 120 and 33 values are below 130, the median belongs to the class-interval 120–130. This class is known as median class.

Median is now calculated by using the formula :

$$\text{Median} = l_1 + \frac{l_2 - l_1}{f}(m - cf)$$

Where l_1 is the lower class limit and l_2 is the upper class limit of the median class , f is the frequency of this class, $m = \frac{n}{2}$ and cf is cumulative frequency upto but not including the median class. $\therefore l_1 = 120, l_2 = 130, m = \frac{50}{2} = 25, f = 20$ and $cf = 13$.

$$\text{Hence median} = 120 + \frac{130 - 120}{20}(25 - 13)$$
$$= 120 + (0{\cdot}5)(12) = 120 + 6 = 126.$$

Similarly, for finding Q_1, the first quartile, we find the Q_1 class (similar to median class), i. e. the class which contains the $\frac{n}{4}$ th observation. Let $q_1 = \frac{n}{4} = \frac{50}{4} = 12{\cdot}5$. Since 5 values are below 110 and 13 values are below 120, 110–120 is the Q_1 class.

The formula for Q_1 is

$$Q_1 = l_1 + \frac{l_2 - l_1}{f}(q_1 - cf)$$

where l_1, l_2 and f are the lower class limit, upper class limit and frequency of the Q_1 class, $q_1 = \frac{n}{4}$ and cf is the cumulative frequency upto but excluding Q_1 class.

$$\therefore l_1 = 110, l_2 = 120, f = 8, q_1 = \frac{50}{4} = 12{\cdot}5, cf = 5$$

$$\therefore Q_1 = 110 + \frac{120-110}{8}(12{\cdot}5-5)$$

$$= 110 + \frac{5}{4}(7{\cdot}5) = 110 + 9{\cdot}375 = 119{\cdot}375.$$

Similary, the Q_3 class is 130–140 since $q_3 = \frac{3n}{4} = 37{\cdot}5$. The formula for Q_3 is

$$Q_3 = l_1 + \frac{l_2 - l_1}{f}(q_3 - cf)$$

where l_1, l_2, f are the lower class limit, upper class limit and frequency of the Q_3 class, cf is the cumulative frequency upto but excluding the Q_3 class and $q_3 = \frac{3n}{4}$.

$$\therefore l_1 = 130, l_2 = 140, f = 10, q_3 = 37{\cdot}5 \text{ and } cf = 33$$

$$\therefore Q_3 = 130 + \frac{140-130}{10}(37{\cdot}5-33)$$

$$= 130 + 4{\cdot}5 = \mathbf{134{\cdot}5}.$$

In this way we can find the median, Q_1 and Q_3 for a continuous grouped frequency distribution.

Note: If a continuous variate is recorded as a discrete variate, the class intervals should be rewritten using class boundaries and we proceed as above, e. g., if we are given the class intervals 20–24, 25–29, 30–34, etc., they should be rewritten as 19·5–24·5, 24·5–29·5, 29·5–34·5, etc. Now since the variate is continuous, the method explained above can be used, where l_1 and l_2 will now mean the corresponding class-boundaries.

§ 12·16. Requirements of a good average:

An average must be representative of the data as a whole. Hence a good average must possess the following desirable properties.

(i) *It should be rigidly defined*: The value of an average must not be left to the mere estimation of the observer but must be calculated using a definite formula.

(*ii*) *It should be based on all observations*: This is because if some observations are left out while calculating an average, it cannot be truly representative of the data as a whole.

(*iii*) *It should be easy to calculate*: The calculations of an average should be easy so that it can be understood by common people with average intelligence, e. g. arithmetic mean is easy to calculate as compared to geometric mean.

(*iv*) *Algebraic treatment of the average must be easily possible* : e. g. if we want to compare two or more series of observations of similar groups then we must be able to calculate the average of the combined data from those of the separate data.

(*v*) *It should be least affected by sampling fluctuations*: If we draw different samples from a population, the average of each sample will have a different value in general. That average which shows less difference in its values for different samples is obviously a better average.

§ 12.17. Merits and demerits of the average :

(i) **Arithmetic mean** : This is the most widely used average. Its *merits* are as follows :

It is a very easy average to understand. It can be easily calculated. It exists for all sets of obrervations. Algebraic manipulations like finding mean of two or more sets of observations from the individual means can be done easily. The mean gives weight to each item in direct proportion to its size. It is based on all values of the variate. It is least affected by sampling fluctuations, i. e. if different samples of the same size are drawn from a population and for each sample, if we calculate mean, median and mode, then means of different samples will show less variation amongst themselves compared to median or mode. The arithmetic mean does not require the arrangement of the data in ascending order as is required for median.

We might even say that arithmetic mean should be used almost everywhere and the other averages should be used only if there is a clear reason for doing so.

The *demerits* of arithmetic mean are as follows :

The arithmetic mean may not be represented in the actual data, when the average age of a class is 17 years, there may not be any student whose age is actually 17 years. The mean cannot be located by inspection as median and mode can. It is affected by unduly large or unduly small items, e. g. the mean of

the values 1, 2, 7, 5, 95 is $\frac{1+2+7+5+95}{5} = 22$ which is hardly representative of the data.

(ii) **Median** : The value of median is not distorted by a few items which are exceptionally large or small. It divides the total number of observations (arranged in order) in exactly two groups which is a useful property in centain cases, e. g. the median of wages of workers is 200 means exactly half of them get less than 200 and half of them more than 200.

The *demerits* of median are that it does not lend itself easily to algebraic treatment. It is not possible to compute the median of 2 or more sets of observations by combining them. It is found out by locating rather than the process of addition, subtraction etc.

(iii) **Mode** : The *merits* are an follows : It is based on all values of the variate and is capable of further algebraic treatment. It can be easily determined from the graph. It is not affected by extreme items.

The *demerits* are that it has little meaning unless there are a large number of observations, it may not exist as there may be many modal values. It is not based on all observations. It cannot be located by a simple arithmetic process and in many cases it is difficult to calculate it precisely.

Hence we can conclude that the mean, which always exists, which can be easily calculated and which is least affected by sampling fluctuations is the best average except when there are a few very large or very small observations.

§ 12·2. Measures of dispersion :

We have studied the measures of location which can be looked upon as central values around which many observations of the data are concentrated. But the individual values of the data will generally differ from this central value by varying amounts. The extent to which they differ from the central value is an important property. This is called dispersion (or spread of data).

Any constant calculated using the given observations so as to give an idea of magnitude of dispersion is called a measure of dispersion.

Let us study some measures of dispersion.

§ 12·21. Mean deviation :

The arithmetic mean of the absolute deviations from the mean of observation is called the *deviation about mean*, i. e. in case of a raw data, if the observations $x_1, x_2, x_3, \ldots, x_n$ have mean $\bar{x}$ then the

$$\text{mean deviation about mean} = \frac{\sum_{i=1}^{n} | x_i - \bar{x} |}{n}$$

In case of a frequency distribution, if the variate x takes values $x_1, x_2, \ldots . x_n$ with frequencies $f_1, f_2, \ldots, f_n$ then the

$$\text{mean deviation about mean} = \frac{\sum_{i=1}^{n} f_i | x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$$

where $\bar{x}$ is the arithmetic mean of x.

In case we are given a grouped frequency distribution, the class marks of the intervals will be taken as value of x_i.

Mean deviation about mean is denoted as *M. D. about mean*.

§ 12·22. Change of origin and scale :

Theorem 12·2 : If the variate x takes values $x_1, x_2, x_3, .., x_n$ with frequencies $f_1, f_2, f_3, .., f_n$ and if we define the variable u by $u_i = \frac{x_i - a}{c}$; $a, c \in R$, $c > 0$ then M. D. of $x = c$ (M. D. of u).

Proof : $\because \quad u_i = \frac{x_i - a}{c} \qquad \therefore \quad x_i = a + cu_i$

Also $\bar{x} = a + c\,\bar{u}$ (Theorem 12·1)

$\therefore \quad x_i - \bar{x} = c\,(u_i - \bar{u})$

$\therefore \quad | x_i - \bar{x} | = c \cdot | u_i - \bar{u} |$

$\therefore \quad f_i | x_i - \bar{x} | = c f_i | u_i - \bar{u} |$

$\therefore \quad \sum_{i=1}^{n} f_i | x_i - \bar{x} | = \sum_{i=1}^{n} c \cdot f_i | u_i - \bar{u} |$

$\therefore \quad \frac{\sum_{i=1}^{n} f_i | x_i - \bar{x} |}{\sum_{i=1}^{n} f_i} = \frac{c \cdot \sum_{i=1}^{n} f_i | u_i - \bar{u}|}{\sum_{i=1}^{n} f_i}$

$\therefore$ M. D. of $x = c \cdot$ (M. D. of u) as required.

§ 12·23. Examples :

Ex. 1. Find the M.D. about mean of the observations given below :

2, 5, 4, 3, 7, 6, 8, 5, 11, 9.

Ans. The mean $\bar{x}$ is given by

$$\bar{x} = \frac{2+5+4+3+7+6+8+5+11+9}{10} = \frac{60}{10} = 6.$$

$\therefore$ M. D. about mean of x

$$= \frac{\left\{\begin{array}{c}|2-6|+|5-6|+|4-6|+|3-6|+|7-6|+|6-6| \\ +|8-6|+|5-6|+|11-6|+|9-6|\end{array}\right\}}{10}$$

$$= \frac{4+1+2+3+1+0+2+1+5+3}{10} = \frac{22}{10} = 2{\cdot}2.$$

Ex. 2. Calculate the mean deviation about mean for the following distribution of the ages of employees in a factory :—

Age of employees	15–25	25–35	35–45	45–55	55–65	65–75
No of employees	40	71	53	22	13	1

Ans.

Age of employees	Class mark x_i	No. of employees f_i	$u_i = \frac{x_i - 40}{10}$	$f_i u_i$
15–25	20	40	−2	−80
25–35	30	71	−1	−71
35–45	40	53	0	0
45–55	50	22	1	22
55–65	60	13	2	26
65–75	70	1	3	3
Total	—	200	—	−151 + 51 = −100

Hence $\Sigma f_i u_i = -100$ and $\Sigma f_i = 200$

$$\therefore \quad \bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i} = \frac{-100}{200} = -0{\cdot}5$$

Now consider

u_i	f_i	$\lvert u_i - \bar{u} \rvert$ where $\bar{u} = -0{\cdot}5$	$f_i \lvert u_i - \bar{u} \rvert$
-2	40	1·5	60·0
-1	71	0·5	35·5
0	53	0·5	26·5
1	22	1·5	33·0
2	13	2·5	32·5
3	1	3·5	3·5
Total	200		191·0

$\therefore \Sigma f_i \lvert u_i - \bar{u} \rvert = 191$ and $\Sigma f_i = 200$

$$\therefore \text{ M. D. of } u = \frac{\Sigma f_i \lvert u_i - \bar{u} \rvert}{\Sigma f_i} = \frac{191}{200} = 0{\cdot}955.$$

Now $u_i = \dfrac{x_i - 40}{10}$, $\quad \therefore \quad c = 10.$

$\therefore$ M. D. of $x = 10$ (M. D. of u)

$\therefore$ M. D. of $x = 10\ (0{\cdot}955) = 9{\cdot}55$

§ 12·24. Variance and standard deviation : (Raw data)

Definition : Variance : If a variate x takes values $x_1, x_2, \ldots x_n$ then the *variance* of x is defined by

$$V(x) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} \quad \text{where } \bar{x} \text{ is the mean of } x.$$

Definition: Standard deviation: If $V(x)$ is the variance of a variate x taking values $x_1, x_2, \ldots, x_n$ then the standard deviation (s. d.) of x is defined as the positive square root of $V(x)$,

i. e., s. d. $= \sqrt{\dfrac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$ where $\bar{x}$ is the mean of x.

Note: Standard deviation of x is also denoted as σ_x where the Greek letter σ (sigma) is used. In this notation the variance of x can be denoted as σ_x^2. The calculation of variance and s. d. for a raw data is done as in the following example.

Ex. Find the variance and s. d. of the set of observations:

2, 5, 4, 3, 7, 6, 8, 5, 11, 9,

Ans. The mean $\bar{x}$ is

$$\frac{2+5+4+3+7+6+8+5+11+9}{10} = 6$$

Variance of x

$$= \frac{1}{10}[(2-6)^2+(5-6)^2+(4-6)^2+(3-6)^2+(7-6)^2$$
$$+(6-6)^2+(8-6)^2+(5-6)^2+(11-6)^2+(9-6)^2]$$
$$= \frac{1}{10}(16+1+4+9+1+0+4+1+25+9)$$
$$= \frac{1}{10}(70) = 7. \qquad \therefore \text{ s. d. of } x = \sqrt{7}.$$

Theorem 12·3: If x takes the values $x_1, x_2, \ldots, x_n$, then

$$V(x) = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2$$

Proof:

$$V(x) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \bar{x}^2 \sum_{i=1}^{n} 1\right]$$

Now $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$ $\quad\therefore\quad \sum_{i=1}^{n} x_i = n\bar{x}$

$$\therefore \quad V(x) = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - 2\bar{x}(n\bar{x}) + \bar{x}^2 n\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 \text{ as required.}$$

§ 12·25. Variance and standard deviation (frequency distribution) :

Definition : Variance : If a variate x takes values $x_1, x_2 \ldots x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively, then the variance of x is defined as

$$V(x) = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i}$$

where $\bar{x}$ is the mean of x

Definition : Standard deviation : If $V(x)$ is the variance of a variate x then σ_x, the standard deviation (s. d.) of x is defined as the positive square root of $V(x)$, i. e. if x takes values $x_1, x_2, x_3 \ldots, x_n$ with frequencies $f_1, f_2, f_3, \ldots, f_n$ respectively, then

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i}}$$

As in theorem 12·3, we put variance in another form in the following theorem.

Theorem 12·4: If x takes $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots f_n$ respectively then

$$V(x) = \frac{\sum_{i=1}^{n} f_i x_i^2}{\sum_{i=1}^{n} f_i} - \left[\frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}\right]^2$$

where $\bar{x}$ is the mean of x.

Proof: Let $\sum_{i=1}^{n} f_i = N$

$$\therefore \quad \text{Variance } V(x) = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2$$

$$= \frac{1}{N}\sum_{i=1}^{n} (f_i x_i^2 - 2f_i x_i \bar{x} + f_i \bar{x}^2)$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i x_i^2 - 2\bar{x} \cdot \sum_{i=1}^{n} f_i x_i + \bar{x}^2 \sum_{i=1}^{n} f_i\right]$$

Now $\bar{x} = \dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i \cdot \quad \therefore \sum_{i=1}^{n} f_i x_i = N\bar{x}.$

$$\therefore \quad V(x) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i^2 - 2\bar{x}(N\bar{x}) + \bar{x}^2(N)\right)$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i x_i^2 - N\bar{x}^2\right]$$

$$= \frac{\sum_{i=1}^{n} f_i x_i^2}{N} - \bar{x}^2$$

$$= \frac{\sum_{i=1}^{n} f_i x_i^2}{\sum_{i=1}^{n} f_i} - \left[\frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}\right]^2 \text{ as required.}$$

Note: To calculate variance using this theorem, we require the values of $\sum_{i=1}^{n} f_i$, $\sum_{i=1}^{n} f_i x_i$ and $\sum_{i=1}^{n} f_i x_i^2$. The first two

sums are calculated as for the calculation of $\bar{x}$ and the third sum is calculated by adding a column $f_i x_i^2$ in the table and adding the figures in it.

The calculation can be further simplified by change of origin and scale. The relevant formula is given in the following theorem.

Theorem 12·5 : If a variate x takes the values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$ respectively and we define a variate u as $u_i = \frac{x_i - a}{c}$ for $i = 1, 2, \ldots, n$ then $V(x) = c^2 . V(u)$.

Proof : We know that

$$V(x) = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i} \quad \text{and} \quad V(u) = \frac{\sum_{i=1}^{n} f_i (u_i - \bar{u})^2}{\sum_{i=1}^{n} f_i}$$

Now $u_i = \frac{x_i - a}{c}$

$\therefore$ $x_i = a + c\, u_i$

Also $\bar{x} = a + c\, \bar{u}$ (Theorem 12·1)

Hence $x_i - \bar{x} = c\,(u_i - \bar{u})$

$\therefore$ $(x_i - \bar{x})^2 = c^2 (u_i - \bar{u})^2$

$\therefore$ $f_i (x_i - \bar{x})^2 = c^2 \cdot f_i (u_i - \bar{u})^2$

$$\therefore \sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = c^2 \sum_{i=1}^{n} f_i (u_i - \bar{u})^2$$

$$\therefore V(x) = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i} = \frac{c^2 \sum_{i=1}^{n} f_i (u_i - \bar{u})^2}{\sum_{i=1}^{n} f_i}$$

$= c^2 V(u)$ as required,

Corollary : S. D. of $x = c \cdot$ (s. d. of u) where $c > 0$, using the notation of theorem 12·4.

The proof is obvious.

Note : (1) If two data relate to the same variate and have approximately equal means then the one with the smaller s. d. is more consistent.

(2) If two distributions of sizes n_1 and n_2 have means m_1 and m_2 and variances v_1 and v_2, then the mean and s. d. of the combined distribution are given by

$$\text{mean} = \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2}$$

$$\text{s. d.} = \sqrt{\frac{n_1 (v_1 + d_1^2) + n_2 (v_2 + d_2^2)}{n_1 + n_2}}$$

where $d_1 = m - m_1$ and $d_2 = m - m_2$, m being the mean of the combined distribution.

§ 12·26. **Examples :**

Ex. 1. The heights of 100 students are given below. Find the mean and s. d. of the distribution :—

Height in inches	60	61	62	63	64	65	66
Number of students	12	28	30	18	6	4	2

Ans.

Height x_i	Number of students f_i	$u_i = x_i - 62$	$f_i u_i$	$f_i u_i^2$
60	12	−2	−24	48
61	28	−1	−28	28
62	30	0	0	0
63	18	1	18	18
64	6	2	12	24
65	4	3	12	36
66	2	4	8	32
Total	100		−2	186

Now $\Sigma f_i = 100$, $\Sigma f_i u_i = -2$, $\Sigma f_i u_i^2 = 186$.

$$\therefore \bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i} = \frac{-2}{100} = -0{\cdot}02$$

$$V(u) = \frac{\Sigma f_i u_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right)^2 = \frac{186}{100} - (-0{\cdot}02)^2$$

$$= 1{\cdot}86 - 0{\cdot}0004 = 1{\cdot}8596$$

Since $x_i = u_i + 62$

$$\therefore \bar{x} = \bar{u} + 62 = -0{\cdot}02 + 62 = 61{\cdot}98$$

and $V(x) = V(u) = 1{\cdot}8596$

$$\therefore \text{S. D. of } x = \sqrt{1{\cdot}8596}$$

Ex. 2. Calculate the mean and s. d. of quantity of grain produced from the following data :

Number of Kgs. of grains produced	No. of plots
200–299	5
300–399	9
400–499	15
500–599	17
600–699	22
700–799	17
800–899	7
900–999	6
1000–1099	2

Ans.

No. of Kgs.	Class mark x_i	f_i	$u_i = \frac{x_i - 649.5}{100}$	$f_i x_i$	$f_i x_i^2$
200–299	249.5	5	−4	−20	80
300–399	349.5	9	−3	−27	81
400–499	449.5	15	−2	−30	60
500–599	549.5	17	−1	−17	17
600–699	649.5	22	0	0	0
700–799	749.5	17	1	17	17
800–899	849.5	7	2	14	28
900–999	949.5	6	3	18	54
1000–1099	1049.5	2	4	8	32
Total :	—	100	—	−37	369

Now $\Sigma f_i = 100$, $\Sigma f_i u_i = -37$, $\Sigma f_i u_i^2 = 369$

$$\therefore \bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i} = \frac{-37}{100} = -0.37$$

$$\text{and } V(u) = \frac{\Sigma f_i u_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right)^2$$

$$= \frac{369}{100} - \left(\frac{-37}{100}\right)^2 = 3.69 - (-0.37)^2$$

$$= 3.69 - 0.1369 = 3.5531$$

$$\therefore \text{ s. d. of } u = \sqrt{3.5531} = 1.885$$

$$\forall\ x_i = 649.5 + 100\, u_i$$

$$\therefore\ \bar{x} = 649.5 + 100\,\bar{u} = 649.5 + 100\,(-0.37)$$

$$= 649.5 - 37 = 612.5$$

$$\text{and s. d. of } x = 100\ (\text{s. d. of } u)$$

$$= 100\ (1.885) = 188.5$$

Ex. 3. A distribution consists of three components with the frequencies of 200, 250 and 300 having the mean of 25, 10 and 15 s. d. s of 3, 4 and 5 respectively. Show that the mean and s. d. for the combined distribution are 16 and 7·2 approximately.

Ans. Mean of combined distribution

$$= \frac{200(25)+250(10)+300(15)}{200+250+300}$$

$$= \frac{5000+2500+4500}{750} = \frac{12000}{750} = 16$$

Now variance of combined distribution

$$\frac{n_1(v_1+d_1^2)+n_2(v_2+d_2^2)+n_3(v_3+d_3^2)}{n_1+n_2+n_3}$$

Here $n_1 = 200,\ v_1 = 9,\quad d_1 = 16-25 = -9$

$n_2 = 250,\ v_2 = 16,\ d_2 = 16-10 = 6$

$n_3 = 300,\ v_3 = 25,\ d_3 = 16-15 = 1$

$\therefore$ Variance of combined distribution

$$= \frac{200(9+81)+250(16+36)+300(25+1)}{200+250+300}$$

$$= \frac{18000+13000+7800}{750} = \frac{38800}{750} = 51{\cdot}73.$$

$\therefore$ S. d. of combined distribution $= \sqrt{51{\cdot}73}$

$= 7{\cdot}2$ approximately.

§ 2·27. Range and Quartile Deviation :

Besides mean deviation and the standard deviation, the other measures of dispersion are range and quartile deviation, i. e. semiinter-quartile range.

Range : The range of a set observations is the difference between the maximum and minimum values of the observation, e. g. for the set of values 1, 16, 4, 9, 12, range $= 16-1 = 15$.

The Quartile Deviation : If Q_1 and Q_3 are the first and the third quartiles of the set of values of a variate x then $(Q_3 - Q_1)$ is called the *inter quartile range* and $\frac{(Q_3-Q_1)}{2}$ is called the semiinter-quartile range. It is also known as the *Quartile Deviation.*

We know that 25% of the values of a variate are below Q_1 and 25% of the values are above Q_3. Hence the inter-quartile range $(Q_3 - Q_1)$ includes the middle 50% of the values, i, e. it is the range of values containing the middle 50% of the values. Quartile deviation is half of the inter-quartile range.

You are already familiar with the method of finding Q_1 and Q_3. With the help of these we can find the quartile deviation.

Ex. Find the interquartile range and quartile deviation of the marks of students for the following frequency distribution of the marks of students.

Marks	0–20	20–40	40–60	60–80	80–100
No. of students	5	6	25	10	4

Ans.

Class Intervals	Frequency	Cumulative Frequency
0–20	5	5
20–40	6	11
40–60	25	36
60–80	10	46
80–100	4	4
	50	

$q_1 = \frac{n}{4} = \frac{50}{4} = 12{\cdot}5$. Hence $40 - 60$ is the Q_1 class.

$$\therefore \quad Q_1 = l_1 + \frac{l_2 - l_1}{f}(q_1 - cf)$$

$$= 40 + \frac{60 - 40}{25} \cdot (12{\cdot}5 - 11)$$

$$= 40 + \frac{20 \times 1{\cdot}5}{25} = 40 + 1{\cdot}2 = 41{\cdot}2$$

$q_3 = \frac{3n}{4} = \frac{3 \times 50}{4} = 37{\cdot}5$. Hence 60 — 80 is the Q_3 class.

$$\therefore \quad Q_3 = l_3 + \frac{l_2 - l_1}{f}(q_3 - cf)$$

$$= 60 + \frac{80 - 60}{10}(37{\cdot}5 - 36)$$

$$= 60 + 2\,(1{\cdot}5) = 60 + 3 = 63.$$

Inter quartile range $= Q_3 - Q_1 = 63 - 41{\cdot}2 = 21{\cdot}8$.

Quartile deviation or semiinter-quartile range

$$= \frac{Q_3 - Q_1}{2} = \frac{21{\cdot}8}{2} = 10{\cdot}9.$$

Exercise 12·3 :

(1) The following data gives the annual crop in kgs. from 10 experimental farms :

2020 2100 2040 2030 2070
2060 2080 2050 2110 2090

Find the mean, mean deviation and s. d. of annual yield.

(2) The following data gives the percentage of the S. S. C. Examination marks of 20 students :

65 72 67 69 70 76 71 73 74 75
71 65 67 73 69 74 66 75 68 70

Find the mean and s. d. of the percentage of marks. What is the range?

(3) Calculate the mean, median, mean deviation and s. d. for each of the following sets of observations :

(a) 20, 50, 40, 30, 70, 60, 80, 50, 110, 90.
(b) 102, 105, 104, 103, 107, 106, 108, 105, 111, 109.
(c) 618, 619, 616, 623, 620, 624, 622, 625, 622, 625, 626, 625.
(d) 1050, 1120, 1070, 1090, 1100, 1060, 1110, 1130, 1140, 1140, 1110, 1050, 1070, 1130 1090, 1140, 1060, 1150, 1080, 1100.

(4) Calculate the mean, median and s. d. for the distribution of the number of children in a sample of 100 families given below :

Number of children	0	1	2	3	4	5	6	7	8
Number of families	6	6	14	23	24	12	8	5	2

(5) Calculate the mean and the mean deviation for the data showing the ages of the employees in a company.

Age of employee :	15–25	25–35	35–45	45–55	55–65	65–75
Number of emplyees	30	101	48	27	13	1

(6) Find the mean, s. d. and the quartile deviation of the following frequency distribution.

Class Interval	*Frequency*
3–4·9	5
5–6·9	8
7–8·9	30
9–10·9	82
11–12·9	45
13–14·9	24
15–16·9	6

(7) Find the mean, s. d. and inter-quartile range of the incomes of 70 workers in a factory from the following data :

Income in Rupees	60–70	70–80	80–90	90–100	100–110	110–120	120–130
Number of workers	6	7	10	20	12	10	5

(8) Calculate the mean and s. d. of the I. Q. of 160 students :

I. Q.	60–70	70–80	80–90	90–100	100–110	110–120	120–130	130–140
Number of students	3	8	12	30	42	36	19	10

(9) The following is the distribution of 300 telephone calls according to their duration in seconds. Calculate mean, s. d. and quartiles.

Duration	*Number of calls*
0–30	8
30–60	15
60–90	41
90–120	79
120–150	82
150–180	45
180–210	23
210–240	5
240–270	2

(10) A frequency distribution after shift of origin and scale is as follows :

u_i	−4	−3	−2	−1	0	1	2	3
f_i	2	5	8	18	22	13	8	4

If the true mean and s. d. are 135·3 and 9·6 respectively, find the frequency distribution in the original form.

(11) A cumulative frequency distribution after the change of origin and scale is as follows :

New variable	−3	−2	−1	0	1	2	3	4
Cumulative frequency	2	7	15	33	55	68	76	80

If the true mean and s. d. are 147·3 and 9·6 respectively, find the frequency distribution in the original form.

(12) The median value of marks (out of 150) obtained by 25 students is 65. It was afterwards found that a student who actually scored 73 marks was wrongly awarded 37 marks and a student who actually scored 48 marks was wrongly awarded 84 marks. What would be the corrected median ? Why ?

(13) The mean and s. d. of 25 values was calculated to be 56 and 2 respectively. It was later discovered that a value copied down as 64 was wrong. What would be the mean and s. d. if this value is omitted ?

(14) In calculating mean and variance of a frequency distribution of 100 observations, the following results were obtained : mean = 9, variance = 19. It was later discovered that one observation 12 was misread as 21. Obtain the mean and s. d. after the correction.

(15) For a frequency distribution of marks in History of 200 candidates (grouped as 0-5, 5-10, 10-15 etc.) the mean and s. d. were found to be 40 and 15. Later a score 53 was corrected to 43. Find the corrected mean and s. d. of the distribution.

(16) The mean and s. d. of a set of 25 items are 127 and 12 respectively. Five items each of value 127 are added to the data. Find the mean and s. d. of the 30 items.

(17) A sample of 25 values has mean 65·2 and s. d. 2·5. Another sample of 75 values from the same population has mean 67·5 and s. d. 2. Find the mean and s. d. of the combined sample of 100 values.

(18) A distribution consists of 3 components with frequencies 50, 60 and 90 having means of 113, 120 and 115 and s. d. 6, 7 and 8 respectively. Find the mean and s. d. of the combined group.

(19) There are two samples of sizes 100 and 150 drawn from the same population. The sample of 100 values has mean 15 and s. d. 3. The whole group of 250 values has mean 15·6 and variance 13·44. Find the mean and s. d. of the second sample of 150 values.

ANSWERS

CHAPTER 2

Exercise 2·1 : (Page 14) : **1.** (a) $M \equiv (3,0), N \equiv (0,5)$, (b) $M \equiv (-3,0), N \equiv (0,4)$, (c) $P \equiv (3,-5)$, (d) $l(OM) = 3, l(MP) = 4, l(OP) = 5$. **2.** 5 sq. units. **3.** 9 sq. units. **4.** (2, 5) and (−2, 5).

Exercise 2·2 : (Page 19) : **1.** (a) 5, (b) $\sqrt{53}$, (c) 10. **2.** $y = 2, -6$. **3.** $a = 5, 6$. **4.** (11, 0), (−5, 0). **5.** (0, 16), (0, −8). **6.** −11. **8.** $a = 3$. **10.** (a) (3, 5), $\sqrt{34}$, (b) (1, 2), 5, (c) $\left(\frac{-3}{2}, \frac{3}{2}\right)$, $\frac{1}{2}\sqrt{130}$, (d) (0, −5), 5. **11.** $a = -2$, 8 and $b = 2, -4$. **12.** $n = 10$ or $n = -8$.

Exercise 2·3 : (Page 30) : **1.** (i) $\left(\frac{1}{3}, 2\right)$; (ii) $\left(\frac{1}{8}, \frac{-23}{4}\right)$; (iii) $\left(\frac{11}{5}, 1\right)$; **2.** (i) (−1, −12); (ii) $\left(\frac{19}{2}, \frac{35}{2}\right)$; (iii) (1, 37). **3.** $\left(0, \frac{17}{3}\right)$, $\left(2, \frac{19}{3}\right)$. **4.** (4, 4), (1, −1), (7, 9). **5.** (13, −4). **6.** (−4, −14). **7.** (−9, 11), (5, −13), (1, 3). **8.** ratio 3 : (−1), $a = \frac{15}{2}$. **9.** x axis : 1 : 2 externally, y axis : 3 : 4 internally **10.** 1 : 2 externally, (0, 3). **11.** $b = -4$, ratio 4 : 3. **12.** $\left(\frac{p^2 - 2pq - q^2}{p - q}, \frac{p^2 + q^2}{p - q}\right)$. **13.** (2, −1). **14.** $B \equiv (6, -9)$ $C \equiv (-6, 9)$.

CHAPTER 3

Exercise 3·1 : (Page 36) : **1.** (a) (2, 0), (b) 11, (c) +2 or −2, (d) $y = 2x$, (e) $x = y + 7$. **2.** $x^2 - 3y^2 = 0$. **3.** $8x^2 + 9y^2 = 0$. **4.** (6, 0), (−2, 0). **5.** $x - 3y + 5 = 0$, point (1, 2) lies on it and (3, −5) does not lie on the locus. **6.** $4x^2 + 4y^2 - 11x - 35y + 74 = 0$. **7.** $x^2 + y^2 - 4x + 4y - 33 = 0$. **8.** $x^2 + y^2 + 8x + 2y + 7 = 0$. **9.** $5x^2 + 5y^2 - 12x - 80y + 140 = 0$. **10.** $2x^2 + 2y^2 - 6x + 8y + 13 = 0$.

Exercise 3·2 : (Page 39) : **1.** $4x^2 + 4y^2 = 25$. **2.** $4x - 9y = 0$. **3.** $8x - 3y = 0$. **4.** $y^2 = 8(x + 1)$. **5.** $x^2 + y^2 + 12x - 4y = 140$. **6.** $25y^2 = 24(5x + 6)$. **8.** $9x + 12y + 11 = 0$, **9.** $9x^2 + 9y^2 - 42x - 42y + 85 = 0$. **10.** $3y = x^2 + 8x + 28$.

Exercise 3·3 : (Page 41) : **1.** (a) $x^2 + y^2 - 4x - 2y - 20 = 0$, (b) $x^2 + y^2 = 16$, (c) $x^2 + y^2 + 4x - 10y + 21 = 0$,

(d) $x^2+y^2+6x+8y+14=0$, (e) centre $\equiv (5,-1)$, radius $=\sqrt{6}$, (f) centre $\equiv (0,5)$, radius $=5$, (g) centre $\equiv (-2,0)$, radius $=6$. 2. $x^2+y^2-6x+2y-159=0$. 3. $x^2+y^2+2x+4y=0$. 4. $x^2+y^2=26$. 5. $x^2+y^2-2x-6y+5=0$. 6. $x^2+y^2-3x-4y=0$. 7. $x^2+y^2-4x-6y=0$.

Exercise 3·4 : (Page 44) : 1. (a) $(x-2, y+3)$, (b) $(1,-1)$, (c) $(-4,4)$, (d) $(0,7)$, (e) $(2,0)$, (f) $(0,0)$, 2 (a) $(x'-4, y'+1)$, (b) $(-2,0)$, (c) $(-8,-2)$, (d) $(0,0)$, (e) $(-4,1)$. 3. $a=-2$ $b=4$. 4. $P\equiv(4,-3)$. 5. $x^2+3y^2+12x-6y+40=0$. 6. $x^2-8x-6y-32=0$. 7. (a) $k=2$, (b) $k=4$, (c) $k=3$, (d) $k=5$, (e) $k=5$.

CHAPTER 4

Exercise 4·1 : (Page 48) : 1. $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$.

2. $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}$.

3. (I) B, B′, B′, B, A′, A′, A, A′ ; (II) B′, B′, B, A. B′, B, A, A′

Exercise 4·2 : (Page 52) :

π : A′ $(-1,0)$; $-1, 0$. $\frac{3\pi}{2}$: B′ $(0,-1)$; $0, -1$.

2π : A $(1,0)$: $1, 0$. $2\pi+\frac{\pi}{2}$: B $(0,1)$; $0, 1$.

4π : A $(1,0)$; $1, 0$. 6π : A $(1,0)$; $1, 0$.

$2n\pi$: A $(1,0)$; $1, 0$. $n\pi$ (n even) : A $(1,0)$; $1, 0$.

3π : A′ $(-1,0)$; $-1, 0$. 5π : A′ $(-1,0)$; $-1, 0$.

$(2n+1)\pi$: A′ $(-1,0)$; $-1, 0$.

$n\pi$ (n odd) : A′ $(-1,0)$; $-1, 0$.

CHAPTER 5

Exercise 5.1 (Page 56) : 1. None of the angles is in standard position. 2. quadrant III, positive ; quadrant II, negative ; quadrant IV, positive ; quadrant IV, negative.

Exercise 5·2 : (Page 65) : 1. 108° ; 420° : 80° ; 855° ; 828°. 2. $x=36$; $y=\frac{11\pi}{6}$. 3. $\frac{5\pi^c}{12}$; 75°. 4. $\frac{7\pi^c}{36}$. 5. C $=\frac{13\pi^c}{18}$; D $=\frac{5\pi^c}{9}$. 6. (i) $\frac{2\pi^c}{3}$; (ii) $\frac{3\pi^c}{5}$. 7. 40° ; 60° ; 80° and $\frac{2\pi^c}{9}$; $\frac{\pi^c}{3}$; $\frac{4\pi^c}{9}$. 8. $\frac{2\pi^c}{9}$; $\frac{\pi^c}{3}$; $\frac{5\pi^c}{9}$; $\frac{8\pi^c}{9}$. 9. $\left(\frac{5\pi+9}{9}\right)^c$; $\left(\frac{5\pi-9}{9}\right)^c$.

Exercise 5·3. (Page 68): 1. $\frac{20\pi''}{3}$; 20π sq. inches. 2. $\frac{5\pi}{3}$ cms. 3. $\frac{15\pi}{2}$ sq. inches. 4. $\frac{3\pi}{2}$ cms.; $\frac{27\pi}{4}$ sq. cms. 5. $8\pi''$; 40π sq. inches. 6. radius $= 9$ cms.; area of sector: $\frac{27\pi}{2}$ sq. cms.; area of circle: 81π sq. cms. 7. $12 + 5\pi$ cms.: 15π sq. cms. 8. 40 cms. 10. 3:2. 11. $\frac{a^2(\pi-2)}{2}$ 12. $\frac{4\pi}{3}$ cms. 13. 2π cms.

Exercise 5·5: (Page 76):

1.	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
(a)	$-\frac{4}{5}$	$-\frac{3}{5}$	$\frac{4}{3}$	$-\frac{5}{4}$	$-\frac{5}{3}$	$\frac{3}{4}$
(b)	$\frac{24}{25}$	$-\frac{7}{25}$	$-\frac{24}{7}$	$\frac{25}{24}$	$-\frac{25}{7}$	$-\frac{7}{24}$
(c)	$-\frac{12}{13}$	$\frac{5}{13}$	$-\frac{12}{5}$	$-\frac{13}{12}$	$\frac{13}{5}$	$-\frac{5}{12}$
(d)	$-\frac{24}{25}$	$-\frac{7}{25}$	$\frac{24}{7}$	$-\frac{25}{24}$	$-\frac{25}{7}$	$\frac{7}{24}$
(e)	$-\frac{12}{13}$	$-\frac{5}{13}$	$\frac{12}{5}$	$-\frac{13}{12}$	$-\frac{13}{5}$	$\frac{5}{12}$
(f)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
(g)	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{3}{4}$
(h)	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
(i)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1

Exercise 5·6: (Page 83): (A) 1. $\frac{-1}{\sqrt{2}}$ 2. $-\sqrt{2}$. 3. -1. 4. $\frac{1}{\sqrt{3}}$. 5. $\frac{-2}{\sqrt{3}}$. 6. 2. 7. $\sqrt{3}$. 8. $-\frac{1}{2}$. (B) 1. 0. 2. (-1). 3. 1. 4. 0.

Exercise 5·7: (Page 86): 1. $\tan\theta = \frac{-3}{4}$; $\cos\theta = -\frac{4}{5}$, $\sec\theta = \frac{-5}{4}$. 2. $\frac{3}{4}$; 3. 8. 4. $\operatorname{cosec}\theta - \cot\theta = -\frac{1}{2}$; $\sec\theta - \tan\theta = -3$. 5. $-\frac{1}{6}$ 6. 0. 7. $\frac{-7}{2}$.

8. $\frac{2\sqrt{3}(\sqrt{3}+1)(\sqrt{3}+\sqrt{2})}{3}$. **9.** 0. **10.** $\frac{9}{7}$.

Exercise 5·9: (**Page 92**): **1.** (a) 1, (b) 1, (c) $\frac{1}{3}$, $-\frac{3}{4}$, (d) 0, $-\frac{1}{2}$. **2.** (a) $-\frac{1}{3}$, (b) $\frac{1}{2}$, (c) solution does not exist. **3.** No, since $\left|k+\frac{1}{k}\right| \geq 2$. **4.** No, since $p^2-4p+6 \geq 2$. **5.** $0 \leqslant k \leqslant 2$.

CHAPTER 6

Exercise 6·1: (**Page 100**): **1.** (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$; (ii) $\frac{\sqrt{3}+1}{2\sqrt{2}}$; (iii) $\frac{\sqrt{3}+1}{2\sqrt{2}}$; (iv) $\frac{1-\sqrt{3}}{2\sqrt{2}}$; **25.** $\tan\frac{\pi}{8}=\sqrt{2}-1$.

Exercise 6·5: (**Page 118**): **1.** From (a) to (f) are periodic and (g) and (h) are not periodic. The fundamental periods are as follows: (a) $\frac{\pi}{2}$, (b) $\frac{2\pi}{7}$, (c) $\frac{\pi}{3}$, (d) π, (e) π, (f) 2π. **2.** (a) $\sin(\pi x)$, (b) $\tan x$, (c) $\cos\left(\frac{x}{2}\right)$, (d) $\sin\left(\frac{2\pi x}{7}\right)$, (e) $\cos\frac{2x}{\pi}$, (f) $\sin\left(\frac{2\pi x}{\pi+1}\right)$.

CHAPTER 7

Exercise 7·1: (**Page 125**): **1.** $m\angle B=\frac{\pi}{4}$; $l(BC)=2\sqrt{2}$.

2. $m\angle B=\frac{\pi}{3}$; $l(AC)=2\sqrt{3}$. **3.** $l(AD)=3\sqrt{3}$. **4.** $l(AD)=\sqrt{3}$, $l(AB)=2\sqrt{3}$.

Exercise 7·2: (**Page 134**): (**B**) **2.** $a=\sqrt{19}$, area $=\frac{3\sqrt{3}}{2}$ sq. cms. **3.** area = 3 sq. cms.; no change in area.

Exercise 7·4: (**Page 137**): **1.** height of hill $=300\sqrt{3}'$; height of tower $=600'$. **2.** $(\sqrt{3}+1)/4$ kms. **3.** $10(\sqrt{3}+1)'$. **4.** $\frac{1}{2}(\sqrt{3}+1)$ miles. **5.** $5(\sqrt{3}+1)'$. **6.** $220(\sqrt{3}-1)$. **7.** height of post $=20\sqrt{3}'$; distance between them $=120'$. **9.** 420·58′. **10.** 200′.

CHAPTER 8

Exercise 8·1 : (Page 145) : **1.** (a) 1, (b) $-\sqrt{3}$, (c) -1, (d) -1, (e) $+1$. **2.** (a) $-\frac{2}{3}$, (b) $\frac{3}{10}$, (c) (3, 0), (d) (0, -7), (e) $\frac{3}{4}$, (f) -3, (g) $\frac{3}{2}$, (h) slope of BC $= -\frac{7}{4}$ and slope of CD $= \frac{4}{7}$. **3.** (c) $a = 8$, (d) $k = \frac{16}{3}$, **4.** (d) $a = -11$. **5.** $-1, \frac{1}{2}$, $\frac{1}{2}, -\frac{7}{10}, -\frac{5}{2}, -2, 2, -2, 2$.

Exercise 8·2 : (Page 147) : **1.** (a) (4, -5), 90°, (b) 90°, $\left(-\frac{5}{4}, \frac{7}{6}\right)$, (c) $y = 0$, (d) $x = 7$, (e) $x = 3$, (f) 4, (g) 2. **2.** (a), (b), (c), (d), (e) and (f) are true. **3.** (a) (i) $y = 0$, (ii) $x = 0$, (b) (i) $y = 5$, (ii) $x = 3$, (iii) $x = 3$, (iv) $y = 5$. **4.** (i) $y = 3$, (ii) $x = 3$, (iii) $x = 3$, (iv) $y = 3$, (v) $y = 8$, $y = -2$, (vi) $y = 8$, $y = -2$. **5.** (i) $y = -2$, (ii) $x = -5$, (iii) $y = 3$, (iv) $x = 5$, (v) $y = 5$. **6.** 90°, $\left(-2, \frac{4}{3}\right)$ **7.** $x = 3$. **8.** (2, 1), (2, 4), (3, 1), (3, 4) and point of intersection of diagonals is $\left(\frac{5}{2}, \frac{5}{2}\right)$.

Exercise 8·3 : (Page 151) : **1.** (a) $y - 1 = 2(x - 0)$, (b) $y + 1 = 3(x + 2)$, (c) $y = 5x$. **2.** (i) $x - y + 3 = 0$, (ii) $x + y + 3 = 0$, (iii) $y = 1$, (iv) $2x - 3y - 6 = 0$. **3.** (i) $3x + y - 5 = 0$. (ii) $2x + y - 1 = 0$, (iii) $x + y - 3 = 0$, (iv) $5x + 3y - 3 = 0$. **4.** $x - y - 2 = 0$. **5.** (i) $x + y + 3 = 0$, (ii) $y = x$ **6.** (i) Eqn. of AC is $x - 3y - 14 = 0$, (ii) Eqn. of BD is $3x + y + 8 = 0$. **7.** (i) $3x - 2y - 1 = 0$, (ii) $2x + 3y - 18 = 0$, (iii) $2x = 3y$. **8** $7x - 2y - 5 = 0$, $4x - y - 3 = 0$, $5x + 2y + 17 = 0$.

Exercise 8·4 : (Page 154) : **1.** (a) $y = 3x + 2$, (b) $5x + y + 2 = 0$, (c) $2x + y + 4 = 0$, (d) $2x - y = 4$, (e) $3x + 2y - 6 = 0$. **2.** $x - y + 7 = 0$. **3.** $x + y = 1$ or $3x - 4y = 24$. **4.** $x + 2y = 6$ or $2x + 5y = 10$. **5.** $3x + 2y = 12$. **6.** $4x + 3y = 12$. **7.** $4x + 3y = 24$.

Exercise 8·5 : (Page 161) : **1.** (a) $3x - 4y + 6 = 0$, (b) $2x$

$-y=11$, (c) $3x+2y=12$, (d) $x-3y=6$, (e) $2x+y=4$, (f) $x+2y=-14$. 2. $x+y=8$, $x-y=2$. 3. $\left(\frac{4}{5}, \frac{13}{5}\right)$. 4. (a) $x-y+2=0$, (b) $2x+y-11=0$, (c) $7x+30y=171$. 5. centroid $\equiv\left(\frac{112}{27}, \frac{16}{9}\right)$; orthocentre $\equiv\left(\frac{19}{9}, \frac{8}{27}\right)$. 6. (i) AD is $x=2$, (ii) BE is $3x-2y=9$, (iii) common point $\equiv\left(2, -\frac{3}{2}\right)$. 7. $k=3$, $x+y=5$. 8. circumcentre $\equiv\left(\frac{7}{2}, 5\right)$, orthocentre $\equiv(2, 3)$. 9. $x-y=7$, $x+4y=12$, $x+2y=10$. 11. $2x-y=0$, (2, 4). 12. (4, 5). 13. $7x-y=38$. 14. $4x-3y=0$, centre $\equiv$ (3, 4). 15. (i) 20 sq. units, (ii) $31x-8y=30$, (iii) $\left(\frac{6}{5}, \frac{9}{10}\right)$. 16. $p=2$.

Exercise 8·6 : (Page 166) : 1. (a) $\frac{\pi}{4}$, (b) $\frac{\pi}{4}$, (c) $\tan^{-1}\left(\frac{3}{5}\right)$, (d) $\tan^{-1}\left(\frac{5}{3}\right)$, (e) $\frac{\pi}{2}$, (f) 0°, (g) $\frac{\pi}{6}$, (h) $\frac{\pi}{3}$. 2. $x-3y=18$, $3x+y=4$. 3. $y-2=(2-\sqrt{3})(x-1)$, $y-2=(2+\sqrt{3})(x-1)$. 4. The equations of the sides are: $x+3y+12=0$, $3x-y+6=0$, $x+3y=8$, $3x-y-14=0$ and equations of diagonals are $x-2y=3$, $2x+y=1$. 5. Vertices : (5, 9), (1, 3) and equations of sides are $5x+y-34=0$, $5x+y-8=0$, $x-5y+14=0$, $x-5y+40=0$.

Exercise 8·7 : (Page 171) : 1. $\frac{1}{13}$ 2. $2\sqrt{10}$. 3. (i) $4\sqrt{5}$, (ii) $\frac{1}{2}$, (iii) $5\sqrt{2}$. 4. (−9, 6), (1, −4). 5. $\left(\frac{16}{11}, \frac{1}{11}\right)$, (1, 1). 6. (5, 0), $\left(-\frac{1}{5}, 0\right)$. 7. $x+3y+22=0$, $x+3y=18$. 8 $3x-4y+30=0$, $3x-4y-10=0$. 9. $6x+6y=5$, $2x-2y=1$. 10. $16y+13=0$, $12x-11=0$.

Exercise 8·8 : (Page 175) : 1. The point lies on the given line itself. 2. The point lies on non-origin side of the given line. 3. for $k<-5$. 4. for $k>-10$. 5. for $k>-2$.

Exercise 8·9 : (Page 177) : 1. 0, ∴ the points are collinear. 2. $\frac{21}{2}$ sq. units. 3. 4 square units. 4. length of perpendicular is $\frac{11}{\sqrt{2}}$, area of triangle is $\frac{33}{2}$ square units.

CHAPTER 9

Exercise 9·1 : (Page 181) : 1. 500. **2.** 0·5 **3.** 0·01. **4.** 40000. **5.** 2. **6.** 432. **7.** 9. **8.** 1296. **9.** 1620000. **10.** 0·001234. **11.** 3456. **12.** 2. **13.** 27. **14.** 46656.

Exercise 9·2 : (Page 189) : 1. 12 sq. mts., $\sqrt{3}$ cu. mts. **2.** 240 sq. mts., 120 cu. mts. **3.** 4680 cu. mts. **4.** $12\sqrt{3}$ cu. mts. **5.** 1800 cu. ft. **6.** 3 cms., 90 sq. cms. **7.** 5000 sq. cms. **8.** $1875\sqrt{3}$ lbs. **9.** (i) areas 1:4 ; volumes 1:8, (ii) areas 1:6 ; volumes 1:18, (iii) areas 1:10 ; volumes 1:20. **10.** $V_1:V_2 :: 8:27$ and $S_1:S_2 :: 4:9$. **11.** Length of edge = 2 cms ; height = 10 cms.

Exercise 9·3 : (Page 193) : 1. 7 mts. **2.** 103 sq. mts., 5 cu. mts. **3.** 200 sq. cms., $\frac{1000\sqrt{3}}{9}$ cu. cms. **4.** 300 sq. cms., $250\sqrt{2}$ cu. cms. **5.** 12 cms. **6.** $600(4-\pi)$ sq. cms. **8.** $S_1:S_2 = 1:9$; $V_1:V_2 = 1:27$. **9.** Length 13 ft., breadth 3 ft., height 9 ft. **10.** 890 sq. ft **11.** 1 cm **12.** breadth 17·5 mts., height 13 mts. **13.** Rs. 5637·5. **14.** 12288.

Exercise 9·4 : (Page 201) : 1. 440 cu. mts. **2.** 9680 lbs., 14 revolutions. **3.** 330 lbs. **4.** 44 cu. mts., 19·36 cu. mts. **5.** 63·8 lbs. **6.** 0·14 cms. **7.** 323·323 kgs., 373·373 kgs. **8.** 210 cu. in. **9.** 12 sq. ft. **10.** (i) 178·2 cu. ft., (ii) 247·5 cu. ft., (iii) 211·2 cu. ft.

CHAPTER 10

Exercise 10·1 (Page 209) : 1. 32 sq. cms. **2.** $144\sqrt{2}$ cu. in **3.** $\frac{4000\sqrt{2}}{3}$ cu. mts. **4.** $\frac{\sqrt{1004}}{3}$ cu. mts. **6.** 2496 cu. ft. **7.** 7600 cu. ft. **8.** $\frac{700}{3}(1+\sqrt{2})$ cu. in.

Exercise 10·2 (Page 217) : **1.** 30π cu. mts. **2.** 292π cu. ft. **3.** 1 : 7 **4** 14 cms., $14\sqrt{3}$ cms. **5.** 22·5 mts. $\frac{5}{2}\sqrt{97}$ mts. **6.** 108π cu. in. **7.** 4 cms, **8.** Conical frustrum will yield 2200 cu. cms. more **9.** 152 gms.. **10.** 15. **11.** 0·5 ft. **12.** 1·6″ **13.** 5·745″.

Exercise 10·3 : (Page 222) : 1. 7776π cu. cms., 1296π sq. cms. **2.** 11088 litres. **4.** $\frac{10}{3}$ inches. **5.** 5. **6.** 1881. **7.** Rs. 33·94. **8.** 155·1 cu. cm., 16·28 kgs.. **9.** 1·06 in. **10.** 1·05″

11. 19 hrs. 12 min. **12.** 3744π cu. cms. **13.** 0·303 cms. **14.** $\frac{484}{30}\pi$ cu. cms. **15.** $\frac{27}{16}$ cms. **16.** 1·302 cms.

CHAPTER 11

Exercise 11·1 : (Page 248) : 4. (*a*), (*c*), (*d*) are continuous variates, (*e*) and (*f*) are discrete variates and (*b*) is an attribute.

5.

Class intervals	*Frequency*
20–24	6
25–29	7
30–34	14
35–39	5
40–44	5
45–49	3
	40

6.

Class Intervals	*Frequency*
20–24	2
25–29	5
30–34	8
35–39	12
40–44	10
45–49	7
50–54	6
	50

7.

Class intervals	*Frequency*
5–10	10
10–15	17
15–20	14
20–25	18
25–30	15
30–35	0
35–40	2
40–45	2
	78

8.

Class intervals	*Frequency*
71–75	4
76–80	3
81–85	4
86–90	5
91–95	6
96–100	3
101–105	15
106–110	2
111–115	2
116–120	1
121–125	2
126–130	1
131–135	1
136–140	1
	50

9.

Class intervals	*Frequency*
0–7	3
8–15	12
16–23	16
24–31	25
32–39	6
40–47	4
48–55	0
56–63	1
	67

10.

Class interval	*Frequency*
60–80	5
80–100	14
100–120	17
120–140	10
140–160	1
160–180	0
180–200	2
200–220	1
	50

Value of x_i	*Cumulative freqnency* (*<Type*) $f<x_i$
60	0
80	5
100	19
120	36
140	46
160	47
180	47
200	49
220	50

11.

Class intervals	*Frequency*
66–80	6
81–95	6
96–110	10
111–125	6
126–140	2
	30

Class boundaries of the interval 81–95 are 80·5 and 95·5 and its class marks is 88. 18 bars have length greater than 95 inches.

12.

x_i	0	10	20	30	40	50	60	70	80	90	100
$f<x_i$	0	3	14	48	95	150	189	212	222	225	226
$f>x_i$	226	223	212	178	131	76	37	14	4	1	0

13.

Class intervals	*Frequency*	x_i	*Cumulative frequency* (<*Type*)
71–75	5	70·5	0
76–80	4	75·5	5
81–85	5	80·5	9
86–90	7	85·5	14
91–95	8	90·5	21
96–100	6	95·5	29
101–105	21	100·5	35
106–110	5	105·5	56
111–115	4	110·5	61
116–120	2	115·5	65
121–125	1	120·5	67
126–130	2	125·5	68
	70	130·5	70

Clase boundaries of class interval 81-85 are 80·5, 85·5.

14.

(a) *Value* x_i	*Cumulative frequency* (<*Type*) $f < x_i$	(b)
0	0	
10	5	(i) 73 students
20	14	(ii) 60 %
30	27	(iii) 19 %
40	55	(iv) 50
50	75	(v) 30
60	87	(vi) 20-50
70	95	(vii) 27 students
80	98	(viii) 13 students
90	99	
100	100	

CHAPTER 12

Exercise 12·1 : (Page 253) :

1.

	Mean	Median	Mode
(a)	2·13	2	1
(b)	4·3	5	5
(c)	2·75	2	2
(d)	12	12	11
(e)	621·6	621	622, 625

2. $k = 5$

Exercise 12·2 : (Page 259): 1. (a) 0, (b) 3·033, (c) 5·4, **2.** 67·9, **3.** Rs. 6360. **4.** Rs. 6·41. **5.** 10·50 **6.** Rs. 117. **7.** 74·5 **8.** $f = 12$. **10.** Males 80 %, females 20%, **11.** number of boys = 25, number of girls = 15.

Exercise 12·3 : (Page 279): 1. Mean = 2065 kgs, Mean Deviation = 25 kgs. and S. D. = 29 kgs. **2.** 70, 3·3, 11 marks. **3.** (a) 60, 22, 26·5; (b) 106, 2·2, 2·65; (c) 626·5, (d) 1084·5, **4.** 3·46, 3, 1·77 children. **5.** 35·23, 1·22 years. **6.** 10·45, 2·41, 1·09. **7.** 95·714, Rs. 15·42 and interquartile range 23·48. 105·9, 1·57, I. Q. **9.** 122·6, 25·97, $Q_1 = 94·125$, $Q_3 = 150$ telphone calls.

10. *Class intervals*	*Frequency*	**11.** *Class intervals*	*Frequency*
109·5–115·5	2	121·5–127·5	2
115·5–121·5	5	127·5–133·5	5
121·5–127·5	8	133·5–139·5	8
127·5–133·5	18	139·5–145·5	18
133·5–139·5	22	145·5–151·5	22
139·5–145·5	13	151·5–157·5	13
145·5–151·5	8	157·5–163·5	8
151·5–157·5	4	163·5–169·5	4

12. The median remains the same. **13.** 55·67, 10·05. **14.** 8·91, 4·19. **15.** 39·95, 14·9. **16.** 127, 10·95. **17.** 69·9, 3·79. **18.** 116, 7, 75. **19.** 16, 4.

MATHEMATICAL TABLES

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

To convert common into Napierian logarithms, multiply by 2·3026.

	0	1	2	3	4	5	6	7	8	9	Mean Differences		
											1 2 3	4 5 6	7 8 9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

To convert common into Napierian logarithms, multiply by 2·3026.

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences		
											1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

NATURAL SINES

Degree	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1′ 2′ 3′	Mean Differences 4′ 5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3 6 9	12 15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3 6 9	12 15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3 6 9	12 15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3 6 9	12 15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3 6 9	12 14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3 6 9	12 14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3 6 9	12 14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3 6 9	12 14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3 6 9	12 14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3 6 9	12 14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3 6 9	11 14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3 6 9	11 14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3 6 9	11 14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3 6 8	11 14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3 6 8	11 14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3 6 8	11 14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3 6 8	11 14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3 6 8	11 14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3 6 8	11 14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3 5 8	11 14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3 5 8	11 14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3 5 8	11 14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3 5 8	11 14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3 5 8	11 14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3 5 8	11 13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3 5 8	11 13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3 5 8	10 13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3 5 8	10 13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3 5 8	10 13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3 5 8	10 13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3 5 8	10 13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2 5 7	10 12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2 5 7	10
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2 5 7	10
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2 5 7	10 1.
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2 5 7	9 12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2 5 7	9 12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2 5 7	9 12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2 5 7	9 11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2 4 7	9 11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2 4 7	9 11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2 4 7	9 11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2 4 6	9 11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2 4 6	8 11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2 4 6	8 10